

Asymptotic expansions of some Toeplitz determinants via the topological recursion

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Abstract: In this article, we study the large n asymptotic expansions of $n \times n$ Toeplitz determinants whose symbols are indicator functions of unions of arc-intervals of the unit circle. In particular, we use an Hermitian matrix model reformulation of the problem to provide a rigorous derivation of the general form of the large n expansion when the symbol is an indicator function of either a single arc-interval or several arc-intervals with a discrete rotational symmetry. Moreover, we prove that the coefficients in the expansions can be reconstructed, up to some constants, from the Eynard-Orantin topological recursion applied to some explicit spectral curves. In addition, when the symbol is an indicator function of a single arc-interval, we provide the corresponding normalizing constants using a Selberg integral and illustrate the theoretical results with numeric simulations up to order $o(\frac{1}{n^4})$. We also briefly discuss the situation when the number of arc-intervals increases with n , as well as more general Toeplitz determinants to which we may apply the present strategy.

1 Introduction: General setting and several reformulations of the problem

1.1 General setting

In this article we are interested in the computation of Toeplitz integrals of the form:

$$Z_n(\mathcal{I}) = \frac{1}{(2\pi)^n n!} \int_{\mathcal{I}^n} d\theta_1 \dots d\theta_n \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2 \quad (1.1)$$

where \mathcal{I} is a union of d ($d \geq 1$) intervals in $[-\pi, \pi]$:

$$\mathcal{I} = \bigcup_{j=1}^d [\alpha_j, \beta_j] \quad \text{with} \quad -\pi \leq \alpha_j \leq \beta_j \leq \pi \quad (1.2)$$

Note that in the case of a full support $\mathcal{I} = [-\pi, \pi]$ it is well known that (See for example appendix A of [4]):

$$Z_n([-\pi, \pi]) = 1 \quad (1.3)$$

Thus, in the rest of the article, **we assume that $\mathcal{I} \neq [-\pi, \pi]$** . Since the integral is obviously invariant under a global angular translation ($\theta \mapsto \theta - \text{Cste}$ 1), we may assume that

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no interval $([\alpha_j, \beta_j])_{1 \leq j \leq d}$ **contains** $\pm\pi$. Integrals of type (1.1) can also be understood as the partition functions of a gas of particles restricted to the set $\mathcal{T} = \{e^{it}, t \in \mathcal{I}\}$ with interactions given by $\prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2$. Moreover, it is also well known that integrals of type (1.1) are Toeplitz integrals that can be reformulated as the determinant of a $n \times n$ Toeplitz matrix with a non-vanishing symbol (defined below) on some arc-intervals of the unit circle. Asymptotic expansions of Toeplitz determinants and integrals of the form (1.1) have been studied for a long time and many results already exist in the literature using different strategies. For example, studies using orthogonal polynomials on the unit circle, properties of powers of random unitary matrices, Fredholm determinants, Riemann-Hilbert problems, etc. have been used to tackle the problem. A non-exhaustive list of articles on the subject is [1, 2, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 29].

The purpose of this article is to provide a rigorous large n expansion of Toeplitz integrals of the form (1.1) **to all order in** $\frac{1}{n}$ using the Eynard-Orantin topological recursion defined in [18]. In particular we shall prove the following results:

- A complete large n expansion when the support is restricted to only one interval (i.e. $d = 1$) in section 2.
- Results up to $O(1)$ when the support is composed of $d = 2r + 1 \geq 3$ intervals of the form $[\alpha_j, \beta_j] = [\frac{2\pi j}{2r+1} - \frac{\pi\epsilon}{2r+1}, \frac{2\pi j}{2r+1} + \frac{\pi\epsilon}{2r+1}]$ with $-r \leq j \leq r$ and $0 < \epsilon < 1$ in section 3.
- Results up to $O(1)$ when the support is composed of $d = 2s \geq 2$ intervals of the form $[\alpha_j, \beta_j] = [\frac{\pi(j-\frac{1}{2})}{s} - \frac{\pi\epsilon}{2s}, \frac{\pi(j-\frac{1}{2})}{s} + \frac{\pi\epsilon}{2s}]$ with $-(s-1) \leq j \leq s$ and $0 < \epsilon < 1$ in section 3.4.
- Partial results when the number of intervals is scaling with n in section 4.

The strategy used in this article is to reformulate the Toeplitz integrals in terms of some Hermitian matrix integrals (with some restrictions on the eigenvalues support). Then we compute the associated spectral curve and the corresponding limiting eigenvalues density. Using the theory developed in [21, 22, 23] we are able to rigorously prove the general form of the large n expansions of the correlators and of the partition function, as well as relate them with quantities computed from the topological recursion. In the case of a single interval, we finally use a Selberg integral to fix the normalization issues of the partition function and thus provide the complete large n expansion of the Toeplitz integrals. We eventually compare our theoretical predictions with numeric simulations performed on the Toeplitz determinant reformulation (which is very convenient for numeric computations) up to $o(\frac{1}{n^4})$.

1.2 Various reformulations of the problem

There are several useful rewritings of the integral (1.1). We list them in the following theorem:

Proposition 1.1 (Various reformulations of the problem) *Defining $\mathcal{I} = \bigcup_{j=1}^d [\alpha_j, \beta_j] \subset (-\pi, \pi)$, $\mathcal{T} = \bigcup_{j=1}^d \{e^{it}, t \in [\alpha_j, \beta_j]\}$ and $\mathcal{J} = \bigcup_{j=1}^d [\tan \frac{\alpha_j}{2}, \tan \frac{\beta_j}{2}]$, the following quantities are equal to each other:*

1. A Toeplitz integral with symbol $f = \mathbb{1}_\tau$:

$$\begin{aligned} Z_n(\mathcal{I}) &= \frac{1}{(2\pi)^n n!} \int_{[-\pi, \pi]^n} d\theta_1 \dots d\theta_n \left(\prod_{k=1}^n f(e^{i\theta_k}) \right) \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2 \\ &= \frac{1}{(2\pi)^n n!} \int_{\mathcal{I}^n} d\theta_1 \dots d\theta_n \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2 \end{aligned} \quad (1.4)$$

2. The determinant of a $n \times n$ Toeplitz matrix:

$$Z_n(\mathcal{I}) = \det (T_{i,j} = t_{i-j})_{1 \leq i,j \leq n} \quad (1.5)$$

with discrete Fourier coefficients given by:

$$\begin{aligned} t_0 &= \frac{1}{2\pi} \sum_{j=1}^d (\beta_j - \alpha_j) = \frac{|\mathcal{I}|}{2\pi} \\ t_k &= \frac{1}{2\pi} \sum_{j=1}^d e^{ik \frac{\alpha_j + \beta_j}{2}} (\beta_j - \alpha_j) \operatorname{sinc} \frac{k(\beta_j - \alpha_j)}{2}, \quad \forall k \neq 0 \end{aligned} \quad (1.6)$$

where we denoted $\operatorname{sinc}(x) = \frac{\sin x}{x}$ the cardinal sine function.

3. A real n -dimensional integral with logarithmic potential and Vandermonde interactions:

$$\begin{aligned} Z_n(\mathcal{I}) &= \frac{2^{n(n-1)}}{(2\pi)^n n!} \int_{\mathcal{I}^n} d\theta_1 \dots d\theta_n \prod_{1 \leq i < j \leq n} \sin^2 \left(\frac{\theta_i - \theta_j}{2} \right) \\ &= \frac{2^{n^2}}{(2\pi)^n n!} \int_{\mathcal{J}^n} dt_1 \dots dt_n \Delta(t_1, \dots, t_n)^2 e^{-n \sum_{k=1}^n \ln(1+t_k^2)} \end{aligned} \quad (1.7)$$

where $\Delta(t_1, \dots, t_n)$ is the usual Vandermonde determinant $\Delta(\mathbf{t}) = \prod_{1 \leq i < j \leq n} (t_i - t_j)^2$.

4. An Hermitian matrix integral with prescribed eigenvalues support:

$$Z_n(\mathcal{I}) = c_n \int_{\mathcal{N}_n(\mathcal{J})} \frac{dM_n}{(\det(I_n + M_n^2))^n} \quad (1.8)$$

where \mathcal{N}_n is the set of Hermitian matrices with eigenvalues prescribed in \mathcal{J} . The normalizing constant c_n is related to the volume of the unitary group:

$$c_n = \frac{1}{(2\pi)^n n!} \frac{1}{\operatorname{Vol} \mathcal{U}_n} = \frac{1}{(2\pi)^n n!} \frac{(n!) \prod_{j=1}^{n-1} j!}{\pi^{\frac{n(n-1)}{2}}} = \frac{1}{2^n \pi^{\frac{n(n+1)}{2}}} \prod_{j=1}^{n-1} j!$$

5. A complex n -dimensional integral over some segments of the unit circle with Vandermonde interactions:

$$Z_n(\mathcal{I}) = (-1)^{\frac{n(n+1)}{2}} i^n \int_{\mathcal{T}^n} du_1 \dots du_n \Delta(u_1, \dots, u_n)^2 e^{-n \sum_{k=1}^n \ln u_k} \quad (1.9)$$

proof:

The proof of the previous theorem is rather elementary. The reformulation in terms of Toeplitz determinant is standard [2, 5]. Indeed it is well known [5] that for a function (usually called “symbol” in the context of Toeplitz integrals) f measurable on the unit circle we have:

$$\begin{aligned} I_n(f) &= \frac{1}{(2\pi)^n n!} \int_{\mathcal{I}^n} d\theta_1 \dots d\theta_n \left(\prod_{k=1}^n f(e^{i\theta_k}) \right) \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2 \\ &= \det (T_{i,j} = t_{i-j})_{1 \leq i,j \leq n} \text{ with } t_k = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{ik\theta} d\theta, \text{ for all } -n \leq k \leq n \end{aligned}$$

Thus, equality between (1.4) and (1.5) corresponds to the application of the last identity with $f = \mathbf{1}_{\mathcal{T}}$. Equality between (1.4) and (1.7) follows from the change of variables $\theta_i = \tan \frac{t_i}{2}$ which is allowed since the support of the angles is included into $(-\pi, \pi)$. With this change of variables we get:

$$|e^{i\theta_i} - e^{i\theta_j}|^2 = \frac{4 \left(\tan \frac{\theta_i}{2} - \tan \frac{\theta_j}{2} \right)^2}{\left(1 + \tan^2 \frac{\theta_i}{2} \right) \left(1 + \tan^2 \frac{\theta_j}{2} \right)}$$

Observing that $d\theta_i$ provides a factor $d\theta_i = \frac{2}{1+t_i^2} dt_i$ and that we have:

$$\prod_{1 \leq i < j \leq n} \frac{1}{(1+t_i^2)(1+t_j^2)} = \prod_{k=1}^n \frac{1}{(1+t_k^2)^{n-1}}$$

immediately gives (1.7). Reformulating the real integral (1.7) in terms of an Hermitian matrix integral is standard (see [17]) from diagonalization $M = U \Lambda U^\dagger$ of normal matrices. We only note here that the support of eigenvalues is prescribed to \mathcal{J} . Eventually the volume of the unitary group can be found in [16] and equality between (1.4) and (1.9) is straightforward from the change of variables $u_i = e^{i\theta_i}$. \square

As presented in the last theorem, the complex (1.9) and real (1.7) integral reformulations of the problem share the important point that the interactions are given by a Vandermonde determinant $\Delta(x)^2$. We stress that this situation is rather exceptional since a (non-affine) change of variables in such integrals does not generally preserve the form of the interactions.

Remark 1.1 *We inform the reader that part of the results proven in this article have already been presented in [4] using the reformulation in terms of the complex integrals (1.9) in the context of return times for the eigenvalues of a random unitary matrix. Indeed, as one can obviously see from (1.7) and (1.9), the complex and real integral reformulations share the crucial fact that the interactions between the eigenvalues are of Vandermonde type: $\Delta(\mathbf{x})^2$. This implies that correlation functions of boths reformulations satisfy some loop equations and that the topological recursion may be applied to both models. This leads to the fact that the spectral curves found in this article are related by some symplectic transformations to the ones presented in [4] and thus that they provide the same sets of free energies (that reconstruct up to some constants $\ln Z_n(\mathcal{I})$). However, we stress that several practical and theoretical issues were disregarded in [4] that can be solved using the real integral reformulation:*

1. In [4], results from [23] were used to justify the form of the large n expansions of the correlators and partition functions. However, results from [23] have only been proved

for integrals with support included in \mathbb{R} , but not in the case of a generic closed curve in \mathbb{C} as would require the unit circle. Though it is believed that the tools developed in [23] should remain valid for certain non-real domains of integration, the rigorous mathematical proof is still missing.

2. In [4], normalization issues related to the partition function were completely disregarded. Consequently, the large n asymptotic expansions of the Toeplitz determinants presented in [4] lack some constant terms that we provide in this paper. Though the normalization issues were not particularly important for the physics problem studied in [4], they become essential when one wants to compute exact probabilities and compare them with numeric simulations. It turns out that the reformulation (1.7) offers, at least in the one interval case, a simple way to deal with the normalization issues by connecting them to a Selberg integral. In the complex setting the connection to a known integral is less obvious and thus the normalization issues were disregarded in [4].
3. Stronger results than those developed for general interactions [23] regarding the large n expansion and the reconstruction by the topological recursion are available in the case of real integrals with Vandermonde interactions like (1.7) in [21, 22, 23]. In particular these results allow a proper rigorous mathematical derivation of the full asymptotic expansion of the Toeplitz integrals (1.1) and provide the explicit expressions of the first orders including the proper normalizing factors.
4. Numerical simulations in [4] were only performed to leading order $O(n^2)$ while we are able to match the theoretical results with numeric simulations up to $O(n^{-6})$ in this article.

2 Study of the one interval case

2.1 Known results

When $d = 1$, we can use the rotation invariance and take $\beta_1 = -\alpha_1 = \pi\epsilon$ with $0 < \epsilon < 1$. For simplicity we shall **denote in this section $a = \tan \frac{\pi\epsilon}{2}$** and a function of ϵ will equivalently be seen as a function of a and vice-versa depending on the relevance of the parameter in the discussion. In this section, we want to compute the large expansion of the Toeplitz integral:

$$Z_n(a) \stackrel{\text{notation}}{\equiv} Z_n(\epsilon) = \frac{1}{(2\pi)^n n!} \int_{[-\pi\epsilon, \pi\epsilon]^n} d\theta_1 \dots d\theta_n \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2 \quad (2.1)$$

The corresponding Toeplitz determinant reformulation is particularly easy:

$$Z_n(\epsilon) = \det (T_{i,j} = t_{i-j})_{1 \leq i,j \leq n} \quad (2.2)$$

with the discrete Fourier coefficients given by:

$$t_0 = \epsilon \text{ and } t_k = \epsilon \sin_c(k\pi\epsilon) \quad , \quad \forall k \neq 0 \quad (2.3)$$

The reformulation in terms of a n -fold integral corresponding to the diagonalized form of an Hermitian matrix integral is given by:

$$Z_n(a) = \frac{2^{n^2}}{(2\pi)^n n!} \int_{[-a,a]^n} dt_1 \dots dt_n \Delta(t_1, \dots, t_n)^2 e^{-n \sum_{k=1}^n \ln(1+t_k^2)} \quad (2.4)$$

Such integrals have been studied by Widom in [2] and do not fall in the standard theory of Toeplitz integrals developed by Szegő. Indeed the standard case corresponds to a symbol f which is strictly positive and continuous on the unit circle. In that case, the standard theory of Toeplitz determinants can be applied and one would obtain the strong Szegő theorem [1]:

$$\frac{1}{n} \ln \det (T_n(f)) \xrightarrow{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \ln(f(e^{i\theta})) d\theta \quad (2.5)$$

When the symbol is discontinuous but remains strictly positive, adaptations of the theory, known as Fisher-Hartwig singularities, have been developed and the convergence of $\frac{1}{n} \ln \det (T_n(f))$ is still obtained though formulas get more involved. However, in our present case, the symbol is vanishing on several intervals of the unit circle and the convergence of $\frac{1}{n} \ln \det (T_n(f))$ does no longer hold. Indeed, in this case, Widom proved the following proposition [2]:

Theorem 2.1 (Widom's result) *Let $0 < \theta_0 < \pi$ and define $\mathcal{T}(\theta_0) = \{e^{it}, t \in [-\theta_0, \theta_0]\}$. Then we have:*

$$\begin{aligned} \ln \det (T_n(\mathbf{1}_{\mathcal{T}(\theta_0)})) &= n^2 \ln \left(\sin \frac{\theta_0}{2} \right) - \frac{1}{4} \ln n - \frac{1}{4} \ln \left(\cos \frac{\theta_0}{2} \right) \\ &\quad + 3\xi'(-1) + \frac{1}{12} \ln 2 + o(1) \end{aligned}$$

where ξ denotes the Riemann ξ -function.

We propose in this section to improve Widom's result by providing a mathematical proof of the form of the large n expansion of $\ln Z_n(\epsilon)$ as well as a general way to compute all sub-leading corrections.

2.2 Spectral curve and limiting eigenvalues density

Integral (2.4) may be seen as a gas of eigenvalues with Vandermonde interactions and evolving in a potential $V(x) = \ln(1 + x^2)$. Consequently, it falls into the category of integrals studied in [21, 22, 23]. In particular, the potential is analytic on \mathbb{R} and has only one minimum at $x = 0$. Moreover, since the support of the integration is restricted to a compact set $[-a, a]$, the convergence issues are trivial. Under such conditions, it is proved in [27, 28] that the empirical eigenvalues density δ_n converges almost surely towards an absolutely continuous limiting eigenvalues density:

$$\delta_n = \frac{1}{n} \sum_{i=1}^n \delta(x - t_i) \xrightarrow{n \rightarrow \infty} d\mu_\infty(x) = \nu(x)dx \quad (2.6)$$

whose support is a finite union of intervals in $[-a, a]$. Determining the limiting eigenvalues density can be done in many ways, but since we plan to use the topological recursion, it seems appropriate to derive the limiting eigenvalues density using the loop equations method. We define the following correlation functions:

Definition 2.1 (Correlation functions) *We define the correlation functions by:*

$$W_{1,a}^{n.c.}(x) = \left\langle \sum_{i=1}^n \frac{1}{x - t_i} \right\rangle$$

$$\begin{aligned}
W_{p,a}^{n.c.}(x_1, \dots, x_p) &= \left\langle \sum_{i_1, \dots, i_p=1}^n \frac{1}{x_1 - t_{i_1}} \cdots \frac{1}{x_p - t_{i_p}} \right\rangle \\
W_{p,a}(x_1, \dots, x_p) &= \left\langle \sum_{i_1, \dots, i_p=1}^n \frac{1}{x_1 - t_{i_1}} \cdots \frac{1}{x_p - t_{i_p}} \right\rangle_c
\end{aligned}$$

where the average of a function of the eigenvalues is defined by:

$$\langle g(t_1, \dots, t_n) \rangle = \frac{2^{n^2}}{(2\pi)^n n! Z_n(a)} \int_{[-a,a]^n} dt_1 \dots dt_n g(t_1, \dots, t_n) \Delta(\mathbf{t})^2 e^{-n \sum_{k=1}^n \ln(1+t_k^2)}$$

The index $_c$ stands for “connected” or “cumulant” in the sense that:

$$\begin{aligned}
W_{1,a}(x) &= W_{1,a}^{n.c.}(x) \\
W_{2,a}(x_1, x_2) &= W_{2,a}^{n.c.}(x_1, x_2) - W_{1,a}^{n.c.}(x_1) W_{1,a}^{n.c.}(x_2) \\
W_{3,a}(x_1, x_2, x_3) &= W_{3,a}^{n.c.}(x_1, x_2, x_3) - W_{1,a}^{n.c.}(x_1) W_{2,a}^{n.c.}(x_2, x_3) - W_{1,a}^{n.c.}(x_2) W_{2,a}^{n.c.}(x_1, x_3) \\
&\quad - W_{1,a}^{n.c.}(x_3) W_{2,a}^{n.c.}(x_1, x_2) + W_{1,a}^{n.c.}(x_1) W_{1,a}^{n.c.}(x_2) W_{1,a}^{n.c.}(x_3) \\
&\text{etc.}
\end{aligned}$$

or in a more general way by the inverse relation:

$$W_{p,a}^{n.c.}(x_1, \dots, x_p) = \sum_{\mu \vdash \{x_1, \dots, x_p\}} \prod_{i=1}^{l(\mu)} W_{|\mu_i|,a}(\mu_i)$$

Integral (2.4) is an Hermitian matrix integral with hard edges at $t = \pm a$. Loop equations for Hermitian matrix integrals with hard edges have been written in many places [24, 25, 26]. They can also be easily obtained with the integral:

$$\frac{1}{(2\pi)^n (n!) Z_n(a)} \int_{[-a,a]^n} dt_1 \dots dt_n \sum_{j=1}^n \frac{d}{dt_j} \left(\frac{1}{x - t_j} \Delta(t_1, \dots, t_n)^2 e^{-n \sum_{i=1}^n \ln(1+t_i^2)} \right) \quad (2.7)$$

Indeed, the last integral is equivalent to:

$$W_{1,a}^2(x) + W_{2,a}(x, x) - \frac{2nx}{1+x^2} W_{1,a}(x) + n \left\langle \sum_{i=1}^n \frac{V'(x) - V'(t_i)}{x - t_i} \right\rangle = \frac{c_1(n)}{x-a} + \frac{c_2(n)}{x+a} \quad (2.8)$$

Equation (2.8) is exact and the coefficients $c_1(n)$ and $c_2(n)$ are given by:

$$\begin{aligned}
c_1(n) &= \frac{e^{-n \ln(1+a^2)}}{(2\pi)^n (n!) Z_n(a)} \sum_{j=1}^n \int_{[-a,a]^{n-1}} \frac{dt_1 \dots dt_{j-1} dt_{j+1} \dots dt_n}{a - t_j} \Delta(t_1, \dots, t_{j-1}, a, t_{j+1}, \dots, t_n)^2 e^{-n \sum_{i \neq j}^n \ln(1+t_i^2)} \\
c_2(n) &= \frac{e^{-n \ln(1+a^2)}}{(2\pi)^n (n!) Z_n(a)} \sum_{j=1}^n \int_{[-a,a]^{n-1}} \frac{dt_1 \dots dt_{j-1} dt_{j+1} \dots dt_n}{a + t_j} \Delta(t_1, \dots, t_{j-1}, -a, t_{j+1}, \dots, t_n)^2 e^{-n \sum_{i \neq j}^n \ln(1+t_i^2)}
\end{aligned} \quad (2.9)$$

Note that since the integral (2.4) is invariant under the change $\mathbf{t} \rightarrow -\mathbf{t}$ we automatically get $c_1(n) + c_2(n) = 0$. In order to obtain the spectral curve of the problem (which is the Stieltjes transform of the limiting eigenvalues density), we take the leading order in n of equation (2.8). Results from [21, 22, 23] show that $W_{1,a}(x) \underset{n \rightarrow \infty}{\sim} n W_{1,a}^{(0)}(x)$ and $W_{2,a}(x_1, x_2) \underset{n \rightarrow \infty}{=} O(1)$.

Defining

$$y(x) = W_{1,a}^{(0)}(x) - \frac{1}{2} V'(x) = W_{1,a}^{(0)}(x) - \frac{x}{1+x^2} \quad (2.10)$$

we end up with:

$$y(x)^2 = \frac{x^2}{(1+x^2)^2} - \frac{2}{1+x^2} \left(\lim_{n \rightarrow \infty} \left\langle \frac{1}{n} \sum_{i=1}^n \frac{1}{1+t_i^2} \right\rangle - x \lim_{n \rightarrow \infty} \left\langle \frac{1}{n} \sum_{i=1}^n \frac{t_i}{1+t_i^2} \right\rangle \right) + \frac{c(a)}{x-a} - \frac{c(a)}{x+a} \quad (2.11)$$

where the constant $c(a)$ is given by $c(a) = \lim_{n \rightarrow \infty} \frac{1}{n} c_1(n)$. Since the integral (2.4) is invariant under $\mathbf{t} \rightarrow -\mathbf{t}$ we get that $\left\langle \sum_{i=1}^n \frac{t_i}{1+t_i^2} \right\rangle = 0$ so that:

$$y(x)^2 = \frac{x^2}{(1+x^2)^2} - \frac{2d(a)}{1+x^2} + \frac{c(a)}{x-a} - \frac{c(a)}{x+a} \quad (2.12)$$

where $d(a)$ and $c(a)$ are so far undetermined constants (i.e. independent of x). Moreover, note that by definition we must have at large x :

$$\begin{aligned} W_{1,a}^{(0)}(x) &= \frac{1}{x} - \lim_{n \rightarrow \infty} \left\langle \frac{1}{n} \sum_{i=1}^n t_i \right\rangle \frac{1}{x^2} + O\left(\frac{1}{x^3}\right) \\ &= \frac{1}{x} + O\left(\frac{1}{x^3}\right) \Rightarrow y(x) = O\left(\frac{1}{x^3}\right) \end{aligned} \quad (2.13)$$

since the integral is invariant under $\mathbf{t} \rightarrow -\mathbf{t}$. Using the fact that $y^2(x) = O\left(\frac{1}{x^6}\right)$ in (2.12) provides two independent equations satisfied by $(c(a), d(a))$ that can be explicitly solved. We find:

$$c(a) = \frac{1}{2a(1+a^2)} \text{ and } d(a) = \frac{2+a^2}{2(1+a^2)} \quad (2.14)$$

Finally, we get:

$$y^2(x) = \frac{1+a^2}{(1+x^2)^2(x^2-a^2)} = \frac{1}{\cos^2(\frac{\pi\epsilon}{2})(1+x^2)^2(x^2-\tan^2(\frac{\pi\epsilon}{2}))} \quad (2.15)$$

In other words since $\cos(\frac{\pi\epsilon}{2}) > 0$ for $\epsilon \in (0, 1)$:

$$y(x) = \frac{1}{\cos(\frac{\pi\epsilon}{2})(1+x^2)\sqrt{x^2-\tan^2(\frac{\pi\epsilon}{2})}} \quad (2.16)$$

This is equivalent to say that the limiting eigenvalues density is given by:

$$d\mu_\infty(x) = \frac{dx}{\pi \cos(\frac{\pi\epsilon}{2})(1+x^2)\sqrt{\tan^2(\frac{\pi\epsilon}{2})-x^2}} \mathbb{1}_{x \in [-\tan \frac{\pi\epsilon}{2}, \tan \frac{\pi\epsilon}{2}]} \quad (2.17)$$

The last density can be verified numerically using Monte-Carlo simulations:



Fig. 1: Empirical eigenvalues density obtained from 100 independent Monte-Carlo simulations of the integral (2.4) in the case $\epsilon = \frac{1}{7}$ and $n = 20$. The black curve is the theoretical curve corresponding to (2.17).

The limiting eigenvalues density is supported on the whole interval $[-\tan \frac{\pi\epsilon}{2}, \tan \frac{\pi\epsilon}{2}]$ and exhibits inverse square-root behavior near the edges of the support:

$$d\mu_\infty(x) \stackrel{x \rightarrow \tan \frac{\pi\epsilon}{2}}{=} O\left(\frac{dx}{\sqrt{x - \tan \frac{\pi\epsilon}{2}}}\right) \text{ and } d\mu_\infty(x) \stackrel{x \rightarrow -\tan \frac{\pi\epsilon}{2}}{=} O\left(\frac{dx}{\sqrt{x + \tan \frac{\pi\epsilon}{2}}}\right) \quad (2.18)$$

Moreover it is strictly positive inside $[-\tan \frac{\pi\epsilon}{2}, \tan \frac{\pi\epsilon}{2}]$.

Remark 2.1 *The spectral curve (2.16) is related to the one found in [4] (eq. C.14) using the complex integral reformulation rather than the real integral reformulation. Indeed, in [4], the spectral curve in the one-interval case is $\tilde{y}^2 = \frac{(\tilde{x}+1)^2}{4\tilde{x}^2(\tilde{x}-e^{i\pi\epsilon})(\tilde{x}+e^{-i\pi\epsilon})}$. In fact, both curves are equivalent up to the symplectic transformation:*

$$\tilde{x} = e^{2i\text{Arctan}(x)} \stackrel{\text{def}}{=} f(x) \text{ and } \tilde{y} = \frac{1}{f'(x)}y = \frac{1+x^2}{2i}e^{-2i\text{Arctan}(x)}y \quad (2.19)$$

This transformation follows from the combination of $x = \tan \frac{\theta}{2}$ and $\tilde{x} = e^{i\theta}$. In particular, as explained in appendix B, both curves provide the same set of “symplectic invariants” $(F^{(g)})_{g \geq 0}$ that reconstruct the large n expansion of $\ln Z_n(a)$ up to some constants. However, the Eynard-Orantin differentials attached to both curves differ which is coherent with the fact that the correlation functions are different in both settings.

Remark 2.2 *If we combine the following changes of variables:*

$$x = \tan \frac{q}{2}, \quad u = \sin \frac{q}{2}, \quad v = \frac{1}{\sin \frac{\pi\epsilon}{2}}u \Leftrightarrow x = \frac{v \sin \frac{\pi\epsilon}{2}}{\sqrt{1 - v^2 \sin^2 \frac{\pi\epsilon}{2}}}$$

then the one form $\omega = ydx$ becomes:

$$\omega = \frac{dv}{\sqrt{v^2 - 1}} \quad (2.20)$$

However, the symplectic invariants of the curve (2.16) are different from the ones generated by the last spectral curve. On the other hand, it may be helpful in the limit $\epsilon \rightarrow 0$ since at leading order it becomes $x = v \frac{\pi\epsilon}{2}$. Therefore, at leading order in $\epsilon \rightarrow 0$, a rescaling $\tilde{y} = \frac{2}{\pi\epsilon}y$ preserves the symplectic form $dx \wedge dy = dv \wedge d\tilde{y}$. As we will see in section 2.4, this is particular helpful since the normalization issues of the partition function can be handled with the help of the symplectic invariants associated to the curve (2.20).

2.3 General form of the large n expansions

As soon as the limiting eigenvalue density (2.17) is determined, we may apply the main results of [21, 22, 23] to obtain the general form of the large n expansions of the correlators and of the partition function. However, we first need to prove that Hypothesis 1.1 of [21] is satisfied so that we may apply the main results of [21]. We have:

Proposition 2.1 *The following conditions (Hypothesis 1.1 of [21]) are met for integral (2.4) (note that in our case, the potential V does not depend on n so that some conditions of [21] are trivially verified):*

- (Regularity): The potential V is continuous on the integration domain $[b_-, b_+]$.
- (Confinement of the potential): If $\pm\infty$ belong to the integration contour, then the potential is assumed to be decaying sufficiently fast:

$$\liminf_{x \rightarrow \pm\infty} \frac{V(x)}{2 \ln |x|} > 1$$

- (One cut regime): The support of the limiting eigenvalues density is a single interval $[\alpha_-, \alpha_+]$ not reduced to a point.
- (Control of large deviations): The function $x \mapsto \frac{1}{2}V(x) + \int_{\mathbb{R}} |x - \xi| d\mu_{\infty}(\xi)$ defined on $[b_-, b_+] \setminus (\alpha_-, \alpha_+)$ achieves its minimum only in α_- or α_+ .
- (Off-Criticality): The limiting eigenvalues density is off-critical in the sense that it is strictly positive inside the interior of its support and behaves like $O\left(\frac{1}{\sqrt{x-b_{\pm}}}\right)$ if b_{\pm} is a hard edge or like $O(\sqrt{x-\alpha_{\pm}})$ if α_{\pm} is a soft edge.
- (Analyticity): V can be extended to an analytic function inside a neighborhood of $[\alpha_-, \alpha_+]$.

proof:

In our case, most of the points required are easily verified:

- (Regularity): $x \mapsto \ln(1+x^2)$ is obviously continuous on $[b_-, b_+] = [-a, a]$.
- (Confinement of the potential): No confinement is required since the support is a compact set of \mathbb{R} .
- (One cut regime): This condition directly follows from equation (2.17).
- (Control of large deviations): Since $\alpha_- = b_- = -a$ and $\alpha_+ = b_+ = a$ (the limiting density is supported on the whole integration domain) then $[b_-, b_+] \setminus (\alpha_-, \alpha_+) = \{\alpha_-, \alpha_+\}$ so the condition is trivially realized.

- (Off-Criticality): We only have two hard edges and equation (2.18) provides the correct behavior. Moreover, we directly observe from its expression that $d\mu_\infty(x)$ is strictly positive inside its support.
- (Analyticity): $x \mapsto \ln(1+x^2)$ is trivially analytic in a neighborhood of $[-a, a]$.

□

Therefore, we can apply the main result of [21] for $\beta = 2$ (as well as theorems 1.3 and 1.4 of [22] or results of [23] for the partition function with hard edges) and we obtain that:

Theorem 2.2 (Large n expansions) *The correlators and the partition functions $Z_n(a)$ admit a large n expansion (usually called “topological expansion”) of the form:*

$$\begin{aligned} W_{p,a}(x_1, \dots, x_p) &= \sum_{g=0}^{\infty} W_{p,a}^{\{2-p-2g\}}(x_1, \dots, x_p) n^{2-p-2g} \\ Z_n(a) &= \frac{n^{n+\frac{1}{4}}}{n!} \exp \left(\sum_{k=-2}^{\infty} \tilde{F}^{\{k\}}(a) n^{-k} \right) \\ \ln Z_n(a) &= -\frac{1}{4} \ln n + \sum_{k=-2}^{\infty} F^{\{k\}}(a) n^{-k} \end{aligned} \quad (2.21)$$

The previous large n expansions have the precise meaning that $\forall K \geq 0$:

$$\begin{aligned} W_{p,a}(x_1, \dots, x_p) &= \sum_{g=0}^K W_{p,a}^{\{2-p-2g\}}(x_1, \dots, x_p) n^{2-p-2g} + o(n^{2-p-2K}) \\ \ln Z_n(a) &= -\frac{1}{4} \ln n + \sum_{k=-2}^K F^{\{k\}}(a) n^{-k} + o(n^{-K}) \end{aligned} \quad (2.22)$$

where the $o(n^{2-p-2K})$ and $o(n^{-K})$ are uniform for x_1, \dots, x_n in any compact set of $[-a, a]$ but are not uniform in n nor K .

Notice that the series expansion of a given correlation function only involves powers of n with the same parity and that the series expansion of $W_{p,a}$ starts at $(\frac{1}{n^{p-2}})$. On the contrary, the large n expansion of the partition function $Z_n(a)$ may involve all powers of n and has an extra factor $n^{n+\frac{1}{4}}$ and $n!$. Indeed, the results of [22, 23] providing the r.h.s. of (2.21) only apply directly to $\frac{(2\pi)^n n!}{2n^2} Z_n(a)$. While, the factors $(2\pi)^n$ and $2n^2$ may be absorbed in the definition of the constants $F^{\{k\}}(a)$, the term $n!$ may not. A direct corollary of the previous theorem is that the coefficients $W_{p,a}^{\{2-p-2g\}}$ of the correlators are obtained from the topological recursion:

Corollary 2.1 (Reconstruction of the correlators via the topological recursion)

For all $p \geq 1$ and $g \geq 0$ we have:

$$W_{p,a}^{\{2-p-2g\}}(x_1, \dots, x_p) dx_1 \dots dx_p = \omega_p^{(g)}(x_1, \dots, x_p) \quad (2.23)$$

where $\omega_p^{(g)}(x_1, \dots, x_p)$ is the (p, g) Eynard-Orantin differential (see appendix B or [18]) computed from the application of the topological recursion to the (genus zero) spectral curve (2.16).

Details about the topological recursion are presented for completeness in appendix B and more can be found in [18] and [19]. In particular since the spectral curve (2.16) is of genus zero, it can be parametrized globally on $\bar{C} = \mathbb{C} \cup \{\infty\}$ via:

$$\begin{aligned} x(z) &= \frac{1}{2} \tan\left(\frac{\pi\epsilon}{2}\right) \left(z + \frac{1}{z}\right) \\ y(z) &= \frac{2}{\sin(\frac{\pi\epsilon}{2}) \left(1 + \frac{1}{4} \tan^2(\frac{\pi\epsilon}{2}) \left(z + \frac{1}{z}\right)\right) \left(z - \frac{1}{z}\right)} \end{aligned} \quad (2.24)$$

Moreover, the normalized bi-differential $\omega_2^{(0)}(z_1, z_2)$ required to initialize the topological recursion is $\omega_2^{(0)}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$ for genus zero curves. The proof of corollary 2.1 is standard. Indeed, by construction the correlations functions $W_{p,a}(x_1, \dots, x_p)$ satisfy the loop equations arising in Hermitian matrix models. Moreover, theorem 2.2 ensures that they have a topological expansion and by definition these correlators may only have singularities at the branchpoints or at the edges. These properties are also satisfied by the Eynard-Orantin differentials (See [18]) and thus both sets must match since, under these conditions (topological expansion and location of the singularities), the loop equations admit a unique solution.

The situation is more complicated for the partition function $Z_n(a)$. Indeed, as discussed in [22] and [23], only $\frac{\partial \ln Z_n(a)}{\partial a}$ can be matched with $\frac{\partial \ln \tau(a)}{\partial a}$ where $\ln \tau$ is the tau-function associated to the spectral curve (2.16) defined by:

$$\ln \tau(a) = - \sum_{g=-1}^{\infty} F_{\text{Top. Rec.}}^{(g+1)}(a) n^{-2g} \quad (2.25)$$

The coefficients $\left(F_{\text{Top. Rec.}}^{(g+1)}(a)\right)_{g \geq -1}$ are computed from the topological recursion and are commonly called “symplectic invariants” or “free energies” (See appendix B or [18] for the formulas). Note that there is a change of convention regarding the indexes between the indexes of the topological recursion (noted $^{(g)}$ and corresponding to n^{-2g+2}) and the indexes of the large n expansion of $Z_n(a)$ (noted $\{k\}$ and corresponding to n^{-k}). We keep the notation of [18] for the topological recursion side to avoid confusion (hence the notation is $F_{\text{Top. Rec.}}^{(0)}$, $F_{\text{Top. Rec.}}^{(1)}$ for powers n^2 , n^0 and so on). Thus, so far we have:

$$\begin{aligned} F^{\{2k+1\}}(a) &= f^{\{2k+1\}} \text{ independent of } a \\ F^{\{2k\}}(a) &= -F_{\text{Top. Rec.}}^{(k+1)}(a) + f^{\{2k\}} \text{ with } f^{\{2k\}} \text{ independent of } a \end{aligned} \quad (2.26)$$

In other words:

$$\ln Z_n(a) = -\frac{1}{4} \ln n + \sum_{k=-1}^{\infty} f^{\{2k+1\}} n^{-2k-1} + \sum_{k=-1}^{\infty} \left(-F_{\text{Top. Rec.}}^{(k+1)}(a) + f^{\{2k\}}\right) n^{-2k} \quad (2.27)$$

For applications in topological string theory and integrable systems, the constants $(f^{(k)})_{k \geq -2}$ are generally disregarded because the normalization of the tau-function is mostly irrelevant. But in the context of Toeplitz determinants and probability one needs to find a way to compute them. Otherwise, one may only consider relative Toeplitz determinants (i.e. ratio of Toeplitz determinants) as studied in [29]. As explained in [22] (section 7) or in [23], a possible strategy is to obtain an exact formula for the partition function for a specific value of the parameters. In our case, this strategy can be carried out and we can relate the limiting case $a \rightarrow 0$ to a Selberg integral. The connection is detailed in appendix A.

2.4 Normalization and computation of the first terms of the expansion

In order to compute $\ln Z_n(\epsilon)$ we need to compute the topological recursion to the spectral curve:

$$\begin{aligned} x(z) &= \frac{1}{2} \tan\left(\frac{\pi\epsilon}{2}\right) \left(z + \frac{1}{z}\right) \\ y(z) &= \frac{2}{\sin\left(\frac{\pi\epsilon}{2}\right) \left(1 + \frac{1}{4} \tan^2\left(\frac{\pi\epsilon}{2}\right) \left(z + \frac{1}{z}\right)\right) \left(z - \frac{1}{z}\right)} \end{aligned} \quad (2.28)$$

The branchpoints of the spectral curve are located at $z = \pm 1$ and we can define a global involution $\bar{z} = \frac{1}{z}$ for which $x(\bar{z}) = x(z)$ and $y(\bar{z}) = -y(z)$. Note that the one-form ydx is given by:

$$ydx(z) = \frac{dz}{z \cos\left(\frac{\pi\epsilon}{2}\right) \left(1 + \frac{1}{4} \tan^2\left(\frac{\pi\epsilon}{2}\right) \left(z + \frac{1}{z}\right)\right)} \quad (2.29)$$

In particular, it is regular at the branchpoints. We now need to compute the first free energies (also called “symplectic invariants”) attached to the spectral curve. Specific formulas presented in [18] are required for $F_{\text{Top. Rec.}}^{(0)}(a)$ and $F_{\text{Top. Rec.}}^{(1)}(a)$ and the corresponding computations are presented in appendix A (equations (A.6) and (A.8)). We find (remind that $a = \tan(\frac{\pi\epsilon}{2})$):

$$\begin{aligned} F_{\text{Top. Rec.}}^{(0)}(\epsilon) &= \ln 2 - \ln\left(\sin\frac{\pi\epsilon}{2}\right) \\ F_{\text{Top. Rec.}}^{(1)}(\epsilon) &= \frac{1}{4} \ln\left(\cos\left(\frac{\pi\epsilon}{2}\right)\right) \\ F_{\text{Top. Rec.}}^{(2)}(\epsilon) &= \frac{1}{64} - \frac{1}{32} \tan^2\left(\frac{\pi\epsilon}{2}\right) \\ F_{\text{Top. Rec.}}^{(3)}(\epsilon) &= -\frac{1}{256} - \frac{1}{128} \tan^2\left(\frac{\pi\epsilon}{2}\right) - \frac{5}{128} \tan^4\left(\frac{\pi\epsilon}{2}\right) \end{aligned} \quad (2.30)$$

Note that we also have $W_{p,a}^{(0)}(x_1, \dots, x_p) = 0$ for all $p \geq 3$ because ydx is regular at the branchpoints. From the definition and the form of the spectral curve, it is also easy to see that the coefficients $\left(W_{p,a}^{(g)}(x_1, \dots, x_p)\right)_{p \geq 1, g \geq 0}$ and $\left(F_{\text{Top. Rec.}}^{(g)}(a)\right)_{g \geq 0}$ are polynomial functions of $a = \tan(\frac{\pi\epsilon}{2})$ and $\sqrt{1+a^2} = \frac{1}{\cos(\frac{\pi\epsilon}{2})}$. In order to determine the constants $(f^{\{k\}})_{k \geq -2}$ we need to match the partition function $Z_n(a)$ with a known case. In our case this can be done with the help of a Selberg integral and is performed in appendix A.3. In the end, we find (denoting ξ the Riemann ξ -function):

$$\begin{aligned} f^{\{-2\}} &= \ln 2 \\ f^{\{-1\}} &= 0 \\ f^{\{0\}} &= 3\xi'(-1) + \frac{1}{12} \ln 2 \\ f^{\{2g\}} &= F^{(g+1)}(a=0) + \frac{4(1-2^{-2g-2})B_{2g+2}}{2g(2g+2)} \text{ for } g \geq 1 \\ f^{\{2k+1\}} &= 0 \text{ for } k \geq 0 \end{aligned} \quad (2.31)$$

In other words, we finally obtain with $a = \tan \frac{\pi\epsilon}{2}$:

$$\ln Z_n(a) = n^2 \ln\left(\sin\left(\frac{\pi\epsilon}{2}\right)\right) - \frac{1}{4} \ln n - \frac{1}{4} \ln\left(\cos\left(\frac{\pi\epsilon}{2}\right)\right) + 3\xi'(-1) + \frac{1}{12} \ln 2$$

$$+ \sum_{g=1}^{\infty} \left(F^{(g+1)}(a=0) - F^{(g+1)}(a) + \frac{4(1-2^{-2g-2})B_{2g+2}}{2g(2g+2)} \right) n^{-2g} \quad (2.32)$$

and the first orders are given by:

$$\begin{aligned} \ln Z_n(a) &= n^2 \ln \left(\sin \left(\frac{\pi\epsilon}{2} \right) \right) - \frac{1}{4} \ln n - \frac{1}{4} \ln \left(\cos \left(\frac{\pi\epsilon}{2} \right) \right) + 3\xi'(-1) + \frac{1}{12} \ln 2 \\ &+ \frac{1}{64n^2} \left(2 \tan^2 \left(\frac{\pi\epsilon}{2} \right) - 1 \right) + \frac{1}{256n^4} \left(1 + 2 \tan^2 \left(\frac{\pi\epsilon}{2} \right) + 10 \tan^4 \left(\frac{\pi\epsilon}{2} \right) \right) \\ &+ O \left(\frac{1}{n^6} \right) \end{aligned} \quad (2.33)$$

Using the rotation invariance we can easily generalize this result for any interval $[\alpha, \beta]$ and we obtain the following theorem:

Theorem 2.3 (Asymptotic expansion of Toeplitz determinants in the one interval case)

For (α, β) such that $0 < |\beta - \alpha| < 2\pi$, the Toeplitz determinant with symbol $f = \mathbb{1}_{\mathcal{T}(\alpha, \beta)}$ where $\mathcal{T}(\alpha, \beta) = \{e^{it}, t \in [\alpha, \beta]\}$ admits a large n expansion of the form (with the same meaning as the one given in theorem (2.2)):

$$\begin{aligned} \ln \det T_n(\mathbb{1}_{\mathcal{T}(\alpha, \beta)}) &= n^2 \ln \left(\sin \left(\frac{|\beta - \alpha|}{4} \right) \right) - \frac{1}{4} \ln n - \frac{1}{4} \ln \left(\cos \left(\frac{|\beta - \alpha|}{4} \right) \right) \\ &+ 3\xi'(-1) + \frac{1}{12} \ln 2 + \sum_{g=1}^{\infty} \left(F^{(g+1)}(a=0) - F^{(g+1)}(a) + \frac{4(1-2^{-2g-2})B_{2g+2}}{2g(2g+2)} \right) n^{-2g} \end{aligned}$$

where $a = \tan \left(\frac{|\beta - \alpha|}{4} \right)$ and the coefficients $(F^{(g)}(a))_{g \geq 2}$ are the Eynard-Orantin free energies (also called symplectic invariants) associated to the spectral curve

$$y^2(x) = \frac{1}{\cos^2 \left(\frac{|\beta - \alpha|}{4} \right) (1+x^2)^2 \left(x^2 - \tan^2 \left(\frac{|\beta - \alpha|}{4} \right) \right)}$$

The previous large n expansion has the precise meaning that $\forall K \geq 1$:

$$\begin{aligned} \ln \det T_n(\mathbb{1}_{\mathcal{T}(\alpha, \beta)}) &= n^2 \ln \left(\sin \left(\frac{|\beta - \alpha|}{4} \right) \right) - \frac{1}{4} \ln n - \frac{1}{4} \ln \left(\cos \left(\frac{|\beta - \alpha|}{4} \right) \right) + 3\xi'(-1) \\ &+ \frac{1}{12} \ln 2 + \sum_{g=1}^K \left(F^{(g+1)}(a=0) - F^{(g+1)}(a) + \frac{4(1-2^{-2g-2})B_{2g+2}}{2g(2g+2)} \right) n^{-2g} + o(n^{-2K}) \end{aligned}$$

In particular the first orders of the expansion are given by:

$$\begin{aligned} \ln \det T_n(\mathbb{1}_{\mathcal{T}(\alpha, \beta)}) &= n^2 \ln \left(\sin \left(\frac{|\beta - \alpha|}{4} \right) \right) - \frac{1}{4} \ln n - \frac{1}{4} \ln \left(\cos \left(\frac{|\beta - \alpha|}{4} \right) \right) \\ &+ 3\xi'(-1) + \frac{1}{12} \ln 2 + \frac{1}{64n^2} \left(2 \tan^2 \left(\frac{|\beta - \alpha|}{4} \right) - 1 \right) \\ &+ \frac{1}{256n^4} \left(1 + 2 \tan^2 \left(\frac{|\beta - \alpha|}{4} \right) + 10 \tan^4 \left(\frac{|\beta - \alpha|}{4} \right) \right) + O \left(\frac{1}{n^6} \right) \end{aligned}$$

Remark 2.3 Another possibility to determine the constants in the large n expansion of $\ln Z_n(\epsilon)$ is to study the limit $\epsilon \rightarrow 1$ (i.e. $a \rightarrow +\infty$) instead of studying the limit $\epsilon \rightarrow 0$

(i.e. $a \rightarrow 0$). Indeed, the transformation $(\tilde{x}, \tilde{y}) = \left(x \tan\left(\frac{\pi\epsilon}{2}\right), \frac{y}{\tan\left(\frac{\pi\epsilon}{2}\right)} \right)$ preserves the symplectic form $dx \wedge dy$. Consequently, the free energies $(F^{(g)}(\epsilon))_{g \geq 0}$ computed from the topological recursion applied to the spectral curve (2.28) are identical to those computed on the curve $y = \frac{\cos\left(\frac{\pi\epsilon}{2}\right)}{(\cos^2\left(\frac{\pi\epsilon}{2}\right) + x^2 \sin^2\left(\frac{\pi\epsilon}{2}\right))\sqrt{x^2-1}}$. Changing $y \rightarrow y \cos\left(\frac{\pi\epsilon}{2}\right)$ does not preserve the free energies of the curve but changes $F^{(g)} \rightarrow (\cos\left(\frac{\pi\epsilon}{2}\right))^{2g-2} F^{(g)}$ (with the special case $F^{(1)} \rightarrow \ln\left(\cos\left(\frac{\pi\epsilon}{2}\right)\right) F^{(1)}$). Thus, we end up with the fact that the free energies $(F^{(g)}(\epsilon))_{g \geq 0}$ of the spectral curve (2.28) satisfy:

$$\lim_{\epsilon \rightarrow 1} (1 - \epsilon)^{2g-2} (1 - (1 - \ln(1 - \epsilon))\delta_{g=1}) F^{(g)}(\epsilon) = \hat{F}^{(g)} : \text{Free energies of the curve } y = \frac{\pi}{2x^2\sqrt{1-x^2}}$$

This curve may be obtained from a real integral on $\mathbb{1}_{(-\infty, 1) \cup (1, +\infty)}$ with Vandermonde interactions and with a potential V given by:

$$V(x) = \left[-\frac{\sqrt{1+x^2}}{x} + \ln\left(x + \sqrt{1+x^2}\right) \right]$$

However, since the free energies of the curve $y = \frac{\pi}{2x^2\sqrt{x^2-1}}$ are not known explicitly to all order (at least to our knowledge) then it does not provide a convenient way to determine the normalizing constants of $\ln(Z_n(\epsilon))$.

2.5 Numerical study

We can efficiently compute the Toeplitz determinants (2.2) numerically up to $n = 35$. This allows to compare the theoretical formula (2.33) with the numeric simulations up to order $o\left(\frac{1}{n^4}\right)$. We obtain the following picture:

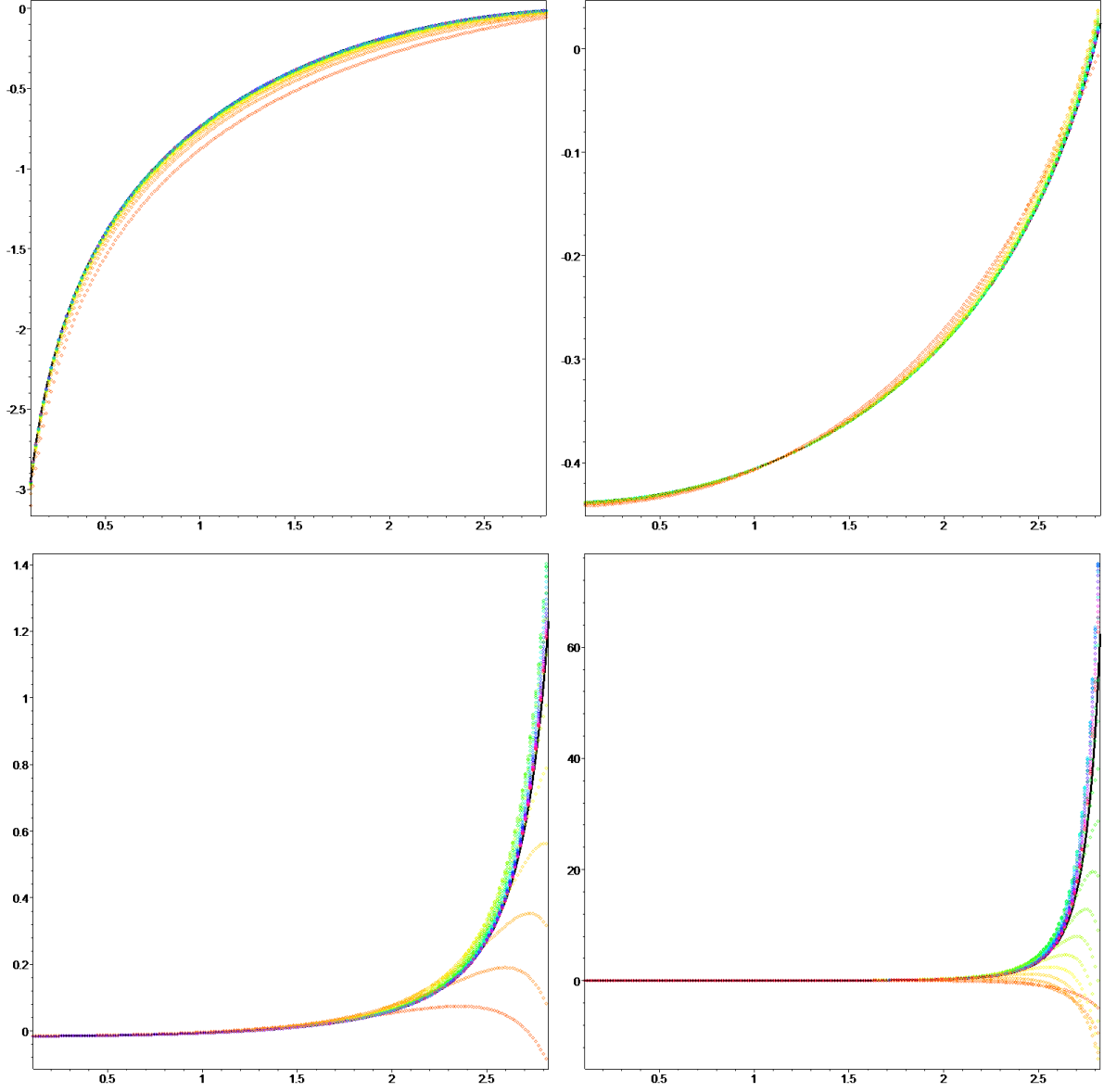


Fig. 2: Computations (from (2.2)) of the Toeplitz determinants $\theta \mapsto \ln Z_n(\frac{\theta}{\pi})$ with $0 < \theta < \pi$ for $2 \leq n \leq 35$ with subtraction of the first coefficients of the large n expansion (2.33) (Colored dots: starting from orange to yellow, green and purple as n increases). The black curves are the theoretical predictions given by (from top to bottom and from left to right): $\theta \mapsto \ln(\sin(\frac{\theta}{2}))$, $\theta \mapsto -\frac{1}{4} \ln(\cos(\frac{\theta}{2})) + 3\xi'(-1) + \frac{1}{12} \ln 2$, $\theta \mapsto \frac{1}{64} (2 \tan^2(\frac{\theta}{2}) - 1)$ and $\theta \mapsto \frac{1}{256} (1 + 2 \tan^2(\frac{\theta}{2}) + 10 \tan^4(\frac{\theta}{2}))$.

We obviously see on the last figure that the numeric simulations are compatible with the theoretical results up to order $o(\frac{1}{n^4})$. This provides additional credit for the general formulas proved in theorem 2.3 and the reconstruction of the expansion from the topological recursion. Note that the reformulation of $Z_n(a)$ in terms of the determinant of a symmetric Toeplitz matrix is particularly useful since it allows fast computations of $Z_n(a)$ even for relatively large values of n . We performed the computations using Maple software and could compute the values of $Z_n(a)$ from $n = 2$ to $n = 35$ in no more than a few minutes on a standard laptop.

3 Toeplitz determinants with a discrete rotational symmetry

Let $r \geq 0$ be a given integer and let $\epsilon \in (0, 1)$ be a given number. In this section we consider the Toeplitz determinants:

$$\begin{aligned} Z_n(\mathcal{I}_r) &= \frac{1}{(2\pi)^n n!} \int_{(\mathcal{I}_r)^n} d\theta_1 \dots d\theta_n \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2 \text{ with} \\ \mathcal{I}_r &= \bigcup_{k=-r}^r \left[\frac{2\pi k}{2r+1} - \frac{\pi\epsilon}{2r+1}, \frac{2\pi k}{2r+1} + \frac{\pi\epsilon}{2r+1} \right] \end{aligned} \quad (3.1)$$

For simplicity, we denote $\alpha_k^{(r)} = \frac{2\pi k}{2r+1} - \frac{\pi\epsilon}{2r+1}$, $\beta_k^{(r)} = \frac{2\pi k}{2r+1} + \frac{\pi\epsilon}{2r+1}$ and $\gamma_k^{(r)} = \frac{2\pi k}{2r+1}$ for $-r \leq k \leq r$. We also define:

$$a_k^{(r)} = \tan\left(\frac{\alpha_k^{(r)}}{2}\right) \text{ and } b_k^{(r)} = \tan\left(\frac{\beta_k^{(r)}}{2}\right) \text{ for } -r \leq k \leq r$$

Note that $Z_n(\mathcal{I}_r)$ can also be interpreted as the probability to obtain all angles $(\theta_j)_{1 \leq j \leq n}$ in \mathcal{I}_r . Similarly to the last section, we also introduce the sets:

$$\begin{aligned} \mathcal{J}_r &= \bigcup_{k=-r}^r [a_k^{(r)}, b_k^{(r)}] = \bigcup_{k=-r}^r \left[\tan\left(\frac{\pi k}{2r+1} - \frac{\pi\epsilon}{2(2r+1)}\right), \tan\left(\frac{\pi k}{2r+1} + \frac{\pi\epsilon}{2(2r+1)}\right) \right] \\ \mathcal{T}_r &= \{e^{i\theta}, \theta \in \mathcal{I}_r\} \end{aligned}$$

Therefore, from theorem 1.1, $Z_n(\mathcal{I}_r)$ can be reformulated as follow:

1. $Z_n(\mathcal{I}_r) = \det T_n^{(r)}$ with $(T_n^{(r)})_{i,j} = t_{i-j}$ the $n \times n$ Toeplitz matrix given by:

$$\begin{aligned} t_0 &= \frac{|\mathcal{I}_r|}{2\pi} = \epsilon \\ t_k &= \epsilon \sin_c\left(\frac{\pi\epsilon k}{2r+1}\right) \delta_{k \equiv 0 [2r+1]} \text{ for } k \neq 0 \end{aligned}$$

Note in particular that the Toeplitz matrices are mostly empty since only bands with indexes multiple of $2r+1$ are non-zero.

2. A real integral with Vandermonde interactions:

$$Z_n(\mathcal{I}_r) = \frac{2^{n^2}}{(2\pi)^n n!} \int_{(\mathcal{J}_r)^n} dt_1 \dots dt_n \Delta(t_1, \dots, t_n)^2 e^{-n \sum_{k=1}^n \ln(1+t_k^2)} \quad (3.2)$$

3. A complex integral:

$$Z_n(\mathcal{I}_r) = (-1)^{\frac{n(n+1)}{2}} i^n \int_{(\mathcal{T}_r)^n} du_1 \dots du_n \Delta(u_1, \dots, u_n)^2 e^{-n \sum_{k=1}^n \ln u_k} \quad (3.3)$$

The situation can be illustrated as follow:

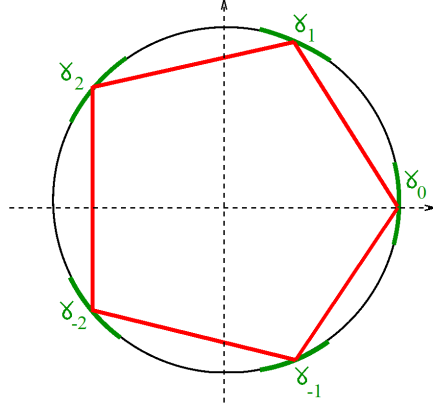


Fig. 3: Illustration (in green) of the set $\mathcal{T}_{r=2}$ for $\epsilon = \frac{1}{5}$.

3.1 Computation of the spectral curve

We want to compute the spectral curve associated to the integral (3.2). This integral corresponds to an Hermitian matrix integral with hard edges at $(a_k^{(r)})_{-r \leq k \leq r}$ and $(b_k^{(r)})_{-r \leq k \leq r}$. Following the same method as in section 2.2, we define $\langle g(\mathbf{t}) \rangle_r$ as the average of the function $g(\mathbf{t})$ relatively to the measure induced by (3.2). With the same arguments as in section 2.2, we get a spectral curve of the form:

$$y(x)^2 = \frac{x^2}{(1+x^2)^2} - \frac{2}{1+x^2} \left(\lim_{n \rightarrow \infty} \left\langle \frac{1}{n} \sum_{i=1}^n \frac{1}{1+t_i^2} \right\rangle_r - x \lim_{n \rightarrow \infty} \left\langle \frac{1}{n} \sum_{i=1}^n \frac{t_i}{1+t_i^2} \right\rangle_r \right) + \sum_{j=-r}^r \left(\frac{A_k}{x - a_k^{(r)}} + \frac{B_k}{x - b_k^{(r)}} \right) \quad (3.4)$$

where the constants A_k (resp. B_k) are given by $A_k = -\lim_{n \rightarrow \infty} \frac{1}{n} A_{k,n}$ and $B_k = -\lim_{n \rightarrow \infty} \frac{1}{n} B_{k,n}$ with:

$$\begin{aligned} A_{k,n} &= -\frac{e^{-n \ln(1+(a_k^{(r)})^2)}}{(2\pi)^n (n!) Z_n(\mathcal{I}_r)} \sum_{j=1}^n \int_{(\mathcal{J}_r)^{n-1}} \frac{dt_1 \dots dt_{j-1} dt_{j+1} \dots dt_n}{a_k^{(r)} - t_j} \Delta(t_1, \dots, t_{j-1}, a_k^{(r)}, t_{j+1}, \dots, t_n)^2 e^{-n \sum_{i \neq j} \ln(1+t_i^2)} \\ B_{k,n} &= \frac{e^{-n \ln(1+(b_k^{(r)})^2)}}{(2\pi)^n (n!) Z_n(\mathcal{I}_r)} \sum_{j=1}^n \int_{(\mathcal{J}_r)^{n-1}} \frac{dt_1 \dots dt_{j-1} dt_{j+1} \dots dt_n}{b_k^{(r)} - t_j} \Delta(t_1, \dots, t_{j-1}, b_k^{(r)}, t_{j+1}, \dots, t_n)^2 e^{-n \sum_{i \neq j} \ln(1+t_i^2)} \end{aligned} \quad (3.5)$$

Note that the integral (3.2) is invariant under $\mathbf{t} \rightarrow -\mathbf{t}$, thus we get that $\left\langle \sum_{i=1}^n \frac{t_i}{1+t_i^2} \right\rangle_r = 0$. Therefore we end up with:

$$y(x)^2 = \frac{x^2}{(1+x^2)^2} - \frac{2d}{1+x^2} + \sum_{j=-r}^r \left(\frac{A_k}{x - a_k^{(r)}} + \frac{B_k}{x - b_k^{(r)}} \right) \quad (3.6)$$

where the coefficients d and $(A_k, B_k)_{-(r-1) \leq k \leq r-1}$ are so far undetermined and independent of x . Observe now that the choice of \mathcal{I}_r implies that we have:

$$a_{-k}^{(r)} = -b_k^{(r)} \text{ for all } 0 \leq k \leq r$$

Using the invariance of the integral relatively to $\mathbf{t} \mapsto -\mathbf{t}$, we obtain that:

$$A_{-k} = -B_k \text{ and } B_{-k} = -A_k \text{ for all } 0 \leq k \leq r \quad (3.7)$$

Consequently we can reduce the spectral curve to:

$$y(x)^2 = \frac{x^2}{(1+x^2)^2} - \frac{2d}{1+x^2} + \sum_{j=-r}^r \frac{2b_k^{(r)} B_k}{x^2 - (b_k^{(r)})^2} \quad (3.8)$$

Similarly to the one interval case, the study of large x implies that $y^2(x) = O\left(\frac{1}{x^6}\right)$ giving some algebraic relations between the $2r+2$ undetermined coefficients (d, B_{-r}, \dots, B_r) . However unlike the one interval case, when $r \geq 1$, these relations are not sufficient to determine completely the coefficients and some additional relations are required. In order to obtain them we perform the following change of variables (denoting $\mathbf{1} = (1, \dots, 1)^t \in \mathbb{R}^n$) in the integral (3.2). Let $-r \leq j_0 \leq r$:

$$\begin{aligned} \mathbf{t} &= \tan\left(\text{Arctan}(\tilde{\mathbf{t}}) + \frac{\pi j_0}{2r+1} \mathbf{e}\right) = \frac{\tilde{\mathbf{t}} + \tan\left(\frac{\pi j_0}{2r+1}\right)}{1 - \tilde{\mathbf{t}} \tan\left(\frac{\pi j_0}{2r+1}\right)} \Leftrightarrow \\ \tilde{\mathbf{t}} &= \tan\left(\text{Arctan}(\mathbf{t}) - \frac{\pi j_0}{2r+1} \mathbf{e}\right) = \frac{\mathbf{t} - \tan\left(\frac{\pi j_0}{2r+1}\right)}{1 + \mathbf{t} \tan\left(\frac{\pi j_0}{2r+1}\right)} \end{aligned} \quad (3.9)$$

Note that the domain of integration \mathcal{J}_r is invariant under the former change of variables since any interval $[a_k^{(r)}, b_k^{(r)}]$ is mapped to $[a_{k-j_0}^{(r)}, b_{k-j_0}^{(r)}]$ where the indexes $k - j_0$ are to be understood modulo $2r+1$. Then, straightforward computations show that:

$$\begin{aligned} dt_i &= \frac{1 + \tan^2\left(\frac{\pi j_0}{2r+1}\right)}{(1 - \tilde{t}_i \tan\left(\frac{\pi j_0}{2r+1}\right))^2} d\tilde{t}_i \\ 1 + t_i^2 &= \frac{(1 + \tan^2\left(\frac{\pi j_0}{2r+1}\right)) (1 + \tilde{t}_i^2)}{(1 - \tilde{t}_i \tan\left(\frac{\pi j_0}{2r+1}\right))^2} \\ (t_i - t_j)^2 &= \frac{(1 + \tan^2\left(\frac{\pi j_0}{2r+1}\right))^2 (\tilde{t}_i - \tilde{t}_j)^2}{(1 - \tilde{t}_i \tan\left(\frac{\pi j_0}{2r+1}\right))^2 (1 - \tilde{t}_j \tan\left(\frac{\pi j_0}{2r+1}\right))^2} \end{aligned} \quad (3.10)$$

Therefore, the general form of the integral remains invariant:

$$\begin{aligned} Z_n(\mathcal{I}_r) &= \frac{2^{n^2}}{(2\pi)^n n!} \int_{(\mathcal{J}_r)^n} dt_1 \dots dt_n \Delta(t_1, \dots, t_n)^2 e^{-n \sum_{k=1}^n \ln(1+t_k^2)} \\ &= \frac{2^{n^2}}{(2\pi)^n n!} \int_{(\mathcal{J}_r)^n} d\tilde{t}_1 \dots d\tilde{t}_n \Delta(\tilde{t}_1, \dots, \tilde{t}_n)^2 e^{-n \sum_{k=1}^n \ln(1+\tilde{t}_k^2)} \end{aligned}$$

and thus the function $W_1(x)$ is given by:

$$\begin{aligned} W_1(x) &= \frac{2^{n^2}}{(2\pi)^n (n!) Z(\mathcal{I}_r)} \int_{(\mathcal{J}_r)^n} d\mathbf{t} \left(\sum_{k=1}^n \frac{1}{x - t_k} \right) \Delta(\mathbf{t})^2 e^{-n \sum_{k=1}^n \ln(1+t_k^2)} = \frac{2^{n^2}}{(2\pi)^n (n!) Z(\mathcal{I}_r)} \\ &\int_{(\mathcal{J}_r)^n} d\tilde{\mathbf{t}} \left(\sum_{k=1}^n \frac{1 - \tilde{t}_k \tan\left(\frac{\pi j_0}{2r+1}\right)}{x - \tan\left(\frac{\pi j_0}{2r+1}\right) - \tilde{t}_k (1 + x \tan\left(\frac{\pi j_0}{2r+1}\right))} \right) \Delta(\tilde{\mathbf{t}})^2 e^{-n \sum_{k=1}^n \ln(1+\tilde{t}_k^2)} \end{aligned} \quad (3.11)$$

We now observe that the equation:

$$\frac{1 - at}{x - a - t(1 + ax)} = \mu + \frac{\nu}{x - t} + \frac{\rho t}{x^2 - t^2} \quad (3.12)$$

where (μ, ν, ρ) are independent of t admits the following solutions:

$$(\mu, \nu, \rho, a) = \left(\frac{a}{1+ax}, \frac{x(1+a^2)}{(x-a)(1+ax)}, -\frac{(1+a^2)(1+x^2)}{(x-a)(1+ax)^2}, -\frac{2x}{x^2-1} \right) \quad (3.13)$$

In particular we always have $\frac{\mu}{1-\nu} = \frac{x}{1+x^2} = \frac{1}{2}V'(x)$. Inserting these results with $a = \tan\left(\frac{\pi j_0}{2r+1}\right)$ and:

$$\begin{cases} x_{j_0} = \tan\left(\frac{\pi i_0}{2r+1} - \frac{\pi}{2(2r+1)}\right) & \text{if } j_0 = 2i_0 \text{ is even} \\ x_{j_0} = \tan\left(\frac{\pi i_0}{2r+1} + \frac{\pi}{2(2r+1)}\right) & \text{if } j_0 = 2i_0 + 1 \text{ is odd} \end{cases} \quad (3.14)$$

that are solutions of $a(x^2 - 1) + 2x = 0$ located outside of \mathcal{J}_r , we end up in (3.11) (using the fact that $\left\langle \sum_{k=1}^n \frac{t}{x^2-t^2} \right\rangle_r = 0$ from the symmetry $\mathbf{t} \rightarrow -\mathbf{t}$) with:

$$W_1(x_{j_0}) = \mu + \nu W_1(x_{j_0}) \Rightarrow W_1(x_{j_0}) = \frac{x_{j_0}}{1+x_{j_0}^2} \Rightarrow y(x_{j_0}) = 0 \quad (3.15)$$

Hence the values $\left(\tan\left(\frac{\pi i}{2r+1} + \frac{\pi}{2(2r+1)}\right)\right)_{-r \leq i \leq r-1}$ (that are identical to $\left(\tan\left(\frac{\pi i}{2r+1} - \frac{\pi}{2(2r+1)}\right)\right)_{-(r-1) \leq i \leq r}$) are $2r$ distinct zeros of the function $x \mapsto y(x)$ located outside \mathcal{J}_r . This provides $2r$ distinct double zeros for $y^2(x)$. From (3.8) and the fact that $y^2(x) = O\left(\frac{1}{x^6}\right)$ as $x \rightarrow \infty$ we get that the spectral curve must be of the form:

$$y^2(x) = \frac{P_{4r+4}(x)}{(1+x^2)^2 \prod_{k=-r}^r (x^2 - (b_k^{(r)})^2)} \quad \text{with } P_{4r+4} \text{ a polynomial of degree } 4r \quad (3.16)$$

Since we have found $4r$ zeros (counted with their multiplicities), we have:

$$y^2(x) = \lambda_r \frac{\prod_{k=0}^{r-1} \left(x^2 - \tan^2\left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)}\right)\right)^2}{(1+x^2)^2 \prod_{k=-r}^r (x^2 - \tan^2\left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)}\right))} \quad (3.17)$$

with the convention that for $r = 0$ we take empty products (like $\prod_{k=0}^{-1} \left(x^2 - \tan^2\left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)}\right)\right)^2$) equal to 1. The constant λ_r can be determined using the behavior of $y^2(x)$ around $\pm i$. Indeed, by definition the function $z \mapsto W_1(z)$ is analytic in a neighborhood of $\pm i$. Consequently, the poles of $y^2(x)$ at $x = \pm i$ only comes from the shift by $-\frac{1}{2}V'(x) = \frac{x}{1+x^2}$. Therefore we should get $y^2(x) \underset{x \rightarrow i}{\sim} \frac{1}{4(x-i)^2}$. We get:

$$\frac{1}{4} = \lambda_r \frac{\prod_{k=0}^{r-1} \left(1 + \tan^2\left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)}\right)\right)^2}{4 \prod_{k=-r}^r \left(1 + \tan^2\left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)}\right)\right)} \Leftrightarrow \lambda_r = \frac{\prod_{k=0}^{r-1} \cos^4\left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)}\right)}{\prod_{k=-r}^r \cos^2\left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)}\right)} \quad (3.18)$$

Finally we get:

$$\begin{aligned}
y^2(x) &= \frac{\prod_{k=0}^{r-1} \cos^4 \left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)} \right)}{\prod_{k=-r}^r \cos^2 \left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right)} \frac{\prod_{k=0}^{r-1} \left(x^2 - \tan^2 \left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)} \right) \right)^2}{(1+x^2)^2 \prod_{k=-r}^r \left(x^2 - \tan^2 \left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right) \right)} \\
&= \frac{\prod_{k=0}^{r-1} \left(x^2 \cos^2 \left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)} \right) - \sin^2 \left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)} \right) \right)^2}{(1+x^2)^2 \prod_{k=-r}^r \left(x^2 \cos^2 \left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right) - \sin^2 \left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right) \right)} \\
&= \frac{\prod_{k=0}^{r-1} \left((x^2 + 1) \cos^2 \left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)} \right) - 1 \right)^2}{(1+x^2)^2 \prod_{k=-r}^r \left((x^2 + 1) \cos^2 \left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right) - 1 \right)} \tag{3.19}
\end{aligned}$$

The corresponding limiting eigenvalues density is therefore given by:

$$\begin{aligned}
d\mu_\infty(x) &= \frac{\prod_{k=0}^{r-1} \cos^2 \left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)} \right)}{\prod_{k=-r}^r \cos \left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right)} \frac{\prod_{k=0}^{r-1} \left| \tan^2 \left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)} \right) - x^2 \right|}{\pi(1+x^2) \prod_{k=-r}^r \sqrt{\left| \tan^2 \left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right) - x^2 \right|}} \mathbb{1}_{\mathcal{J}_r}(x) dx \\
&= \frac{\prod_{k=0}^{r-1} \left| (1+x^2) \cos^2 \left(\frac{\pi k}{2r+1} + \frac{\pi}{2(2r+1)} \right) - 1 \right|}{\pi(1+x^2) \prod_{k=-r}^r \sqrt{\left| (1+x^2) \cos^2 \left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right) - 1 \right|}} \mathbb{1}_{\mathcal{J}_r}(x) dx \tag{3.20}
\end{aligned}$$

In particular, one can verify that it is properly normalized: $\int_{\mathcal{J}_r} d\mu_\infty(x) = 1$. Note that in the case $r = 0$, we recover the spectral curve for a single interval (2.16). We can verify numerically the last limiting eigenvalues density with Monte-Carlo simulations:

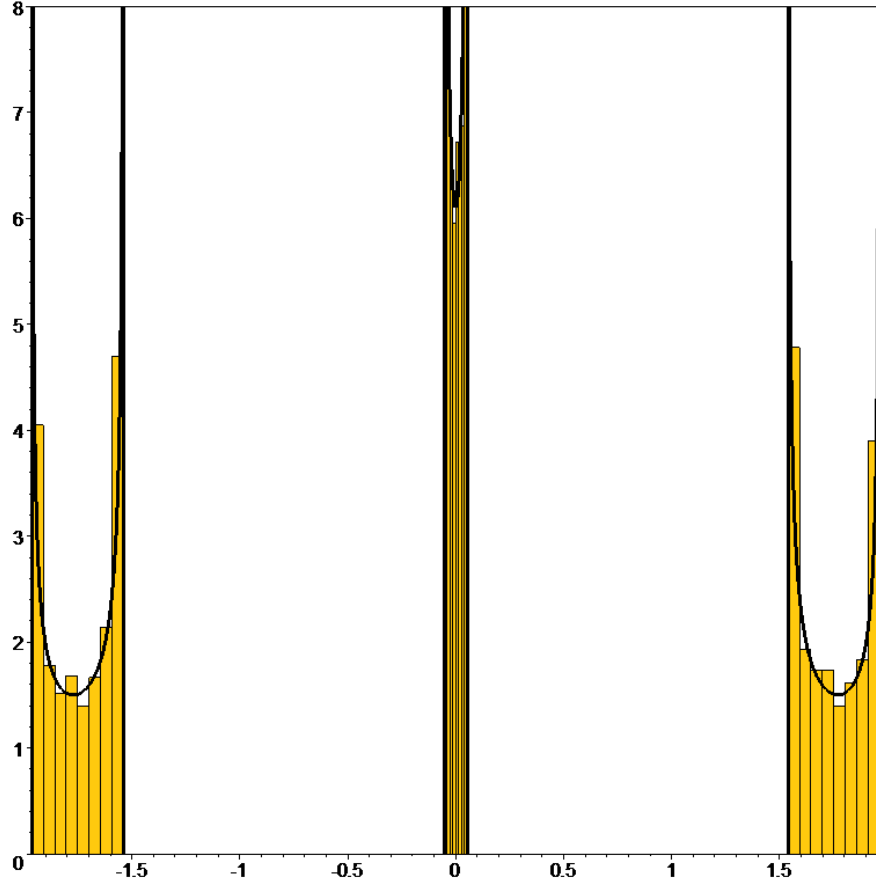


Fig. 4: Histogram of 50 independent simulations of the eigenvalues density induced by (3.2) in the case $r = 1$, $\epsilon = \frac{1}{10}$ and $n = 60$. The black curve is the theoretical limiting eigenvalues density computed in equation (3.20)

For a clearer view, it is also interesting to zoom on each interval:

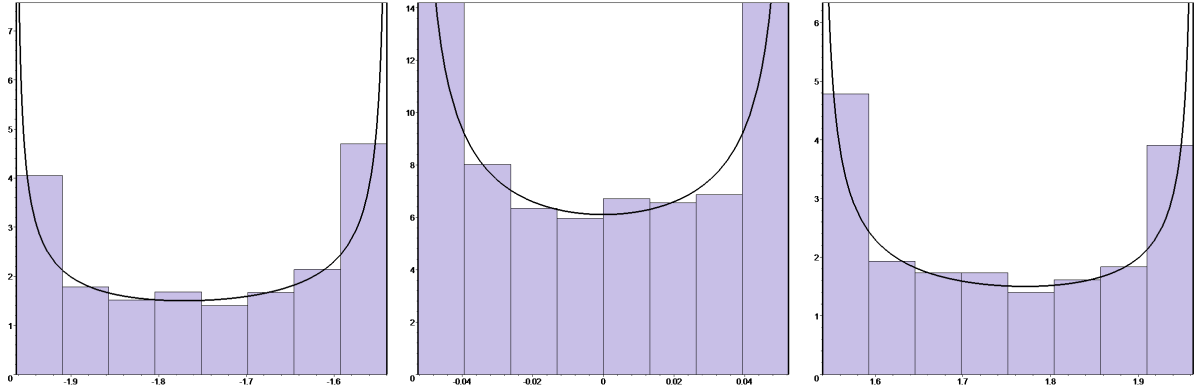


Fig. 5: Zoom of the previous histogram in each of the 3 intervals. The black curve is the theoretical limiting eigenvalues density computed in equation (3.20).

3.2 Filling fractions

We want to determine the proportion of eigenvalues in each of the $2r + 1$ intervals of the limiting eigenvalues density. We start with the previous result:

$$ydx = \lambda_r \frac{\prod_{i=-r}^{r-1} \left(x - \tan \left(\frac{i\pi}{2r+1} + \frac{\pi}{2(2r+1)} \right) \right)}{(1+x^2) \prod_{k=-r}^r \sqrt{\left(x - \tan \left(\frac{\pi k}{2r+1} - \frac{\pi\epsilon}{2(2r+1)} \right) \right) \left(x - \tan \left(\frac{\pi k}{2r+1} + \frac{\pi\epsilon}{2(2r+1)} \right) \right)}} dx \quad (3.21)$$

and observe that the filling fraction corresponding to the k_0^{th} interval, with $-r \leq k_0 \leq r$, is given by:

$$\epsilon_{k_0+r+1} \stackrel{\text{def}}{=} \frac{1}{i\pi} \int_{\tan\left(\frac{\pi k_0}{2r+1} - \frac{\pi\epsilon}{2(2r+1)}\right)}^{\tan\left(\frac{\pi k_0}{2r+1} + \frac{\pi\epsilon}{2(2r+1)}\right)} d\mu_\infty(x) \quad (3.22)$$

Let $-r \leq j_0 \leq r$ be a given integer with $j_0 \neq 0$. We perform the change of variables $x = \frac{\tilde{x} - \tan\left(\frac{\pi j_0}{2r+1}\right)}{1 + \tilde{x} \tan\left(\frac{\pi j_0}{2r+1}\right)}$ described above in the previous integral. Observe in particular that we have $\frac{dx}{1+x^2} = \frac{d\tilde{x}}{1+\tilde{x}^2}$. Moreover under this change of variable, the interval $x \in \left[\tan\left(\frac{\pi k}{2r+1} - \frac{\pi\epsilon}{2(2r+1)}\right), \tan\left(\frac{\pi k}{2r+1} + \frac{\pi\epsilon}{2(2r+1)}\right) \right]$ is directly mapped to the interval $\tilde{x} \in \left[\tan\left(\frac{\pi(k+j_0)}{2r+1} - \frac{\pi\epsilon}{2(2r+1)}\right), \tan\left(\frac{\pi(k+j_0)}{2r+1} + \frac{\pi\epsilon}{2(2r+1)}\right) \right]$. We also have the identities:

$$\begin{aligned} \left(x - \tan \left(\frac{\pi k}{2r+1} \pm \frac{\pi\epsilon}{2(2r+1)} \right) \right) &= \frac{\left(\tilde{x} - \tan \left(\frac{\pi(k+j_0)}{2r+1} \pm \frac{\pi\epsilon}{2(2r+1)} \right) \right) \left(1 - \tan \left(\frac{\pi k}{2r+1} \pm \frac{\pi\epsilon}{2(2r+1)} \right) \tan \left(\frac{\pi j_0}{2r+1} \right) \right)}{1 + \tilde{x} \tan \left(\frac{\pi j_0}{2r+1} \right)} \\ \left(x - \tan \left(\frac{i\pi}{2r+1} + \frac{\pi}{2(2r+1)} \right) \right) &= \frac{\left(\tilde{x} - \tan \left(\frac{(i+j_0)\pi}{2r+1} + \frac{\pi}{2(2r+1)} \right) \right) \left(1 - \tan \left(\frac{\pi i}{2r+1} + \frac{\pi}{2(2r+1)} \right) \tan \left(\frac{\pi j_0}{2r+1} \right) \right)}{1 + \tilde{x} \tan \left(\frac{\pi j_0}{2r+1} \right)} \\ &\quad \text{if } i + j_0 \not\equiv r[2r+1] \end{aligned} \quad (3.23)$$

When $i_0 + j_0 \equiv r[2r+1]$ then we get instead:

$$\begin{aligned} \left(x - \tan \left(\frac{i_0\pi}{2r+1} + \frac{\pi}{2(2r+1)} \right) \right) &= \frac{-\tan\left(\frac{\pi j_0}{2r+1}\right) - \tan\left(\frac{i_0\pi}{2r+1} + \frac{\pi}{2(2r+1)}\right)}{1 + \tilde{x} \tan\left(\frac{\pi j_0}{2r+1}\right)} \\ &= \frac{-\tan\left(\frac{\pi j_0}{2r+1}\right) - \frac{1}{\tan\left(\frac{\pi j_0}{2r+1}\right)}}{1 + \tilde{x} \tan\left(\frac{\pi j_0}{2r+1}\right)} \\ &= \frac{-1}{\cos\left(\frac{\pi j_0}{2r+1}\right) \sin\left(\frac{\pi j_0}{2r+1}\right) (1 + \tilde{x} \tan\left(\frac{\pi j_0}{2r+1}\right))} \end{aligned} \quad (3.24)$$

Observe now that the powers of $1 + \tilde{x} \tan\left(\frac{\pi j_0}{2r+1}\right)$ produce a factor with power 1 at the numerator and that we can express it like:

$$1 + \tilde{x} \tan\left(\frac{\pi j_0}{2r+1}\right) = \tan\left(\frac{\pi j_0}{2r+1}\right) \left(\tilde{x} - \tan\left(\frac{(r+j_0)\pi}{2r+1} + \frac{\pi}{2(2r+1)}\right) \right) \quad (3.25)$$

Therefore collecting the terms depending on \tilde{x} we get:

$$\prod_{k=-r}^r \sqrt{\left(\tilde{x} - \tan\left(\frac{\pi(k+j_0)}{2r+1} - \frac{\pi\epsilon}{2(2r+1)}\right) \right) \left(\tilde{x} - \tan\left(\frac{\pi(k+j_0)}{2r+1} + \frac{\pi\epsilon}{2(2r+1)}\right) \right)}$$

$$\begin{aligned}
&= \prod_{k=-r}^r \sqrt{\left(\tilde{x} - \tan\left(\frac{\pi k}{2r+1} - \frac{\pi \epsilon}{2(2r+1)} \right) \right) \left(\tilde{x} - \tan\left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right) \right)} \\
&\quad \left(\tilde{x} - \tan\left(\frac{(r+j_0)\pi}{2r+1} + \frac{\pi}{2(2r+1)} \right) \right) \prod_{i=-r, i+j_0 \not\equiv r[2r+1]}^{r-1} \left(\tilde{x} - \tan\left(\frac{(i+j_0)\pi}{2r+1} + \frac{\pi}{2(2r+1)} \right) \right) \\
&= \prod_{i=-r, i+j_0 \not\equiv r[2r+1]}^r \left(\tilde{x} - \tan\left(\frac{(i+j_0)\pi}{2r+1} + \frac{\pi}{2(2r+1)} \right) \right) \\
&= \prod_{i=-r}^{r-1} \left(\tilde{x} - \tan\left(\frac{i\pi}{2r+1} + \frac{\pi}{2(2r+1)} \right) \right) \tag{3.26}
\end{aligned}$$

Thus we obtain:

$$\epsilon_{k_0+r+1} = C_{j_0} \epsilon_{k_0+j_0+r+1} \tag{3.27}$$

where the constant C_{j_0} is given by:

$$\begin{aligned}
C_{j_0} &= - \frac{\prod_{i=-r, i+j_0 \not\equiv r[2r+1]}^{r-1} \left(1 - \tan\left(\frac{\pi i}{2r+1} + \frac{\pi}{2(2r+1)} \right) \tan\left(\frac{\pi j_0}{2r+1} \right) \right)}{\cos^2\left(\frac{\pi j_0}{2r+1} \right) \prod_{k=-r}^r \sqrt{\left(1 - \tan\left(\frac{\pi k}{2r+1} - \frac{\pi \epsilon}{2(2r+1)} \right) \tan\left(\frac{\pi j_0}{2r+1} \right) \right) \left(1 - \tan\left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right) \tan\left(\frac{\pi j_0}{2r+1} \right) \right)}} \\
&= - \frac{\prod_{i=-r, i+j_0 \not\equiv r[2r+1]}^{r-1} \frac{\cos\left(\frac{\pi(i+j_0)}{2r+1} + \frac{\pi}{2(2r+1)} \right)}{\cos\left(\frac{\pi i}{2r+1} \right) \cos\left(\frac{\pi j_0}{2r+1} \right)}}{\cos^2\left(\frac{\pi j_0}{2r+1} \right) \prod_{k=-r}^r \sqrt{\frac{\cos\left(\frac{\pi(k+j_0)}{2r+1} + \frac{\pi \epsilon}{2r+1} \right) \cos\left(\frac{\pi(k+j_0)}{2r+1} - \frac{\pi \epsilon}{2r+1} \right)}{\cos\left(\frac{\pi k}{2r+1} - \frac{\pi \epsilon}{2r+1} \right) \cos\left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2r+1} \right) \cos^2\left(\frac{\pi j_0}{2r+1} \right)}}} \\
&= 1 \tag{3.28}
\end{aligned}$$

Hence we have just proved that all filling fractions are equal:

Proposition 3.1 *The filling fractions are the same in each interval:*

$$\forall 1 \leq k \leq 2r+1 : \epsilon_k = \frac{1}{2r+1}$$

Remark 3.1 *This result is compatible with the one found in [4] where the filling fractions were also proved to be equal.*

3.3 General form of the large n asymptotic

After determining the limiting eigenvalues density, the next step is to determine the general form of the asymptotic of the correlation functions and of the partition function. However, since the spectral curve is of strictly positive genus ($\mathbf{g} = 2r$), the general form of the asymptotic is more complicated than in the one interval case. Using the main result of [22] or [23] we can prove the following:

Theorem 3.1 *We have the following large n expansion:*

$$\begin{aligned}
Z_n(\mathcal{I}_r) &= \frac{n^{n+\frac{1}{4}(2r+1)}}{n!} \exp\left(\sum_{k=-2}^{\infty} n^{-k} F_{\epsilon^*}^{\{k\}} \right) \\
&\quad \left\{ \sum_{m \geq 0} \sum_{\substack{l_1, \dots, l_m \geq 1 \\ k_1, \dots, k_m \geq -2 \\ \sum_{i=1}^m (l_i + k_i) > 0}} \frac{n^{-\left(\sum_{i=1}^m (l_i + k_i) \right)}}{m!} \left(\bigotimes_{i=1}^m \frac{F_{\epsilon^*}^{\{k_i\}, (l_i)}}{l_i!} \right) \cdot \nabla_{\nu}^{\otimes \left(\sum_{i=1}^m l_i \right)} \right\} \Theta_{-n\epsilon^*} \left(\mathbf{0} | F_{\epsilon^*}^{\{-2\}, (2)} \right)
\end{aligned}$$

(3.29)

where Θ is the Siegel theta function:

$$\Theta_{\gamma}(\nu, \mathbf{T}) = \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp \left(-\frac{1}{2}(\mathbf{m} + \gamma) \cdot \mathbf{T} \cdot (\mathbf{m} + \gamma) + \nu \cdot (\mathbf{m} + \gamma) \right)$$

and $F_{\epsilon}^{\{2k\},(l)}$ are defined as the l^{th} derivative of the coefficient $F_{\epsilon}^{\{2k\}}$ relatively to the filling fractions $\epsilon = (\epsilon_1, \dots, \epsilon_{2r+1})^T \in \{\mathbf{u} \in (\mathbb{Q}_+)^{2r+1} / \sum_{i=1}^n u_i = 1\}$. The “optimal” filling fractions ϵ^* corresponds to the vector of filling fractions (i.e. the proportion of eigenvalues in each interval of the support of the limiting eigenvalues density) of the limiting eigenvalues density:

$$(\epsilon^*)_{r+1+k} = \int_{\tan\left(\frac{\pi k}{2r+1} - \frac{\pi \epsilon}{2(2r+1)}\right)}^{\tan\left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)}\right)} d\mu_{\infty}(x) \text{ for all } -r \leq k \leq r$$

Again large n expansions presented in this theorem are to be understood as asymptotic expansions up to any arbitrary large negative power of n as in theorem (2.2).

The proof of the last theorem consists in verifying the conditions required to apply the main theorem of [22] and [23]. The conditions are very similar to the one-interval case but for completeness we summarize them here:

- (Regularity): The potential V is continuous on the integration domain $\bigcup_{k=-r}^r [b_{-}^{(k)}, b_{+}^{(k)}]$. In our case $x \mapsto \ln(1+x^2)$ is obviously continuous on \mathbb{R} .
- (Confinement of the potential): Not required since the integration domain is a compact set of \mathbb{R} .
- (Genus $2r$ regime): The support of the limiting eigenvalues density is given by the union of $2r+1$ single intervals $[\alpha_{-}^{(k)}, \alpha_{+}^{(k)}]$ not reduced to a point. This is trivial from the explicit expression (3.20).
- (Control of large deviations): The function $x \mapsto \frac{1}{2}V(x) + \int_{\mathbb{R}} |x - \xi| d\mu_{\infty}(\xi)$ defined on $\left(\bigcup_{k=-r}^r [b_{-}^{(k)}, b_{+}^{(k)}] \right) \setminus \left(\bigcup_{k=-r}^r (\alpha_{-}^{(k)}, \alpha_{+}^{(k)}) \right)$ achieves its minimum only at the endpoints $\alpha_{-}^{(k)}$ or $\alpha_{+}^{(k)}$. In our case this condition is trivial since the limiting eigenvalues density spans the whole integration domain.
- (Off-Criticality): The limiting eigenvalues density is off-critical in the sense that it is strictly positive inside the interior of its support and behaves like $O\left(\frac{1}{\sqrt{x-b_{\pm}}}\right)$ if b_{\pm} is a hard edge or like $O(\sqrt{x-\alpha_{\pm}})$ if α_{\pm} is a soft edge. In our case, we have $2(2r+1)$ hard edges and the explicit expression (3.20) provides the correct behavior. Moreover it is obvious from (3.20) that $d\mu_{\infty}(x)$ is strictly positive inside its support.
- (Analyticity): V can be extended into an analytic function inside a neighborhood of the integration domain. In our case, $x \mapsto \ln(1+x^2)$ is obviously analytic in a neighborhood of any compact set of \mathbb{R} .

As one can see, the general form of the large n expansion in the multi-cut regime is much more complicated than in the one-cut regime since eigenvalues may move from an interval to another and thus a summation on the filling fractions is necessary and adds new terms (given by the Siegel Theta function). The term $\exp\left(\sum_{k=-1}^{\infty} n^{-2k} F_{\epsilon^*}^{\{2k\}}\right)$ is usually called the “perturbative” or “formal” part of the expansion. It is connected to the symplectic invariants computed from the topological recursion by the following proposition (See [22] for details):

Proposition 3.2 *The coefficients $\left(F_{\epsilon^*}^{\{k\}}\right)_{k \geq -2}$ are related to the symplectic invariants $(F^{(g)})_{g \geq 0}$ computed from the topological recursion applied to the spectral curve (3.19) by:*

$$\begin{aligned} \forall k \geq -1 & : F_{\epsilon^*}^{\{2k\}} = -F^{(2k+2)} + f_{2k} \text{ with } f_{2k} \text{ independent of } \epsilon \\ \forall k \geq -1 & : F_{\epsilon^*}^{\{2k+1\}} = f_{2k+1} \text{ with } f_{2k+1} \text{ independent of } \epsilon \end{aligned}$$

Note that the non-perturbative part of (3.29) starts at $O(1)$. The main difficulty of the expansion (3.29) lies in the fact that the non-perturbative part requires the knowledge of the spectral curve in the case where the filling fractions are arbitrarily fixed. In our case, if we take arbitrary filling fractions, the symmetries (3.15) are lost and thus the determination of the spectral curve requires to solve fixed filling fractions conditions:

$$\epsilon_k = \oint_{\mathcal{A}_k} d\mu_{\infty}(x) \text{ for all } -r \leq k \leq r$$

with \mathcal{A}_k a closed contour circling the interval $\left[\tan\left(\frac{\pi k}{2r+1} - \frac{\pi \epsilon}{2(2r+1)}\right), \tan\left(\frac{\pi k}{2r+1} + \frac{\pi \epsilon}{2(2r+1)}\right)\right]$. Unfortunately, solving analytically these conditions remains an open challenge and therefore the non-perturbative part of (3.29) remains mostly out of reach for theoretical computations.

Finally, we also note that even the perturbative part is challenging in the multi-cut regime. Indeed, proposition 3.2 determines the perturbative part up to some constants but, unlike the one-cut regime, these normalization issues are not easy to solve. Indeed, the standard way is to choose the parameters (in our case ϵ) in such a way that the initial integral is explicitly connected to a known case. In the one cut regime, we connected the integral to a Selberg integral after a proper rescaling. However in the multi-cut case, there is no obvious connection to any known integral and the determination of the normalizing constants remains an open problem. The only known cases are $F^{\{-2\}}$ and $F^{\{-1\}}$ (given in section 1.4 of [22] with the important reminder that we included a factor $\frac{2^{n^2}}{(2\pi)^n n!}$ in front of the partition function that is absent in [22]) as well as the coefficient in front of $\ln n$. We get:

$$\begin{aligned} F^{\{-2\}} &= \ln 2 - F^{(0)} \Leftrightarrow f_{-2} = \ln 2 \\ F^{\{-1\}} &= 0 \Leftrightarrow f_{-1} = 0 \end{aligned}$$

We end up with:

$$\begin{aligned} \ln(Z_n(\mathcal{I}_r)) = \ln \det T_n^{(r)} &= (\ln 2 - F^{(0)}) n^2 - \frac{2r+1}{4} \ln n + O(1) \\ &= \frac{n^2}{2r+1} \ln\left(\sin\left(\frac{\pi \epsilon}{2}\right)\right) - \frac{2r+1}{4} \ln n + O(1) \end{aligned} \quad (3.30)$$

The next orders of the large n expansion, and in particular the $O(1)$ term, exhibit a far more complex structure than in the one-cut case. Indeed, theorem 3.29 and the fact that the

present situation has a discrete symmetry (in particular $\epsilon^* = (\frac{1}{2r+1}, \dots, \frac{1}{2r+1})$) imply that the next orders depend on the remainder of the Euclidean division of n by $2r+1$. In fact, as discussed in [22], for a given remainder m such that $0 \leq m \leq 2r$, the sequence $(Z_{(2r+1)p+m})_{p \in \mathbb{N}}$ admit a large p expansion taking a similar form as the one-cut case with n replaced by p (the precise form being given in theorem 2.2) and with coefficients depending on the remainder m . Nevertheless, reconstructing the $F^{\{0\}}$ term for a given remainder m is still challenging because it implies to compute some derivatives (evaluated at $\epsilon^* = (\frac{1}{2r+1}, \dots, \frac{1}{2r+1})$) of the coefficient $F^{\{-2\}}$ relatively to the filling fractions and compute the asymptotic of the Siegel Θ function as described in theorem 3.1. However, because we could compute numerically the Toeplitz determinants up to sufficiently large n , we can propose some conjectures for the coefficient $F^{\{0\}}$ depending on the value of the remainder m .

Conjecture 3.1 (Conjecture for $F^{\{0\}}$) *We conjecture the following:*

- In the case $m = 0$, $F^{\{0\}}$ has a dependence in ϵ given by $-\frac{2r+1}{4} \ln \left(\cos \left(\frac{\pi\epsilon}{2} \right) \right) + C_0$ where C_0 is a constant. This generalizes the one-cut case that would correspond to $r = 0$.
- The cases $m = j$ and $m = 2r - 1 - j$ for some $1 \leq j \leq r$ provides the same $F^{\{0\}}$ coefficient. However, the dependence in ϵ is more involved than for $m = 0$ and we conjecture that it is given by:

$$F^{\{0\}}(j) = A_j \ln \left(\cos \left(\frac{\pi\epsilon}{2} \right) \right) + B_j \ln \left(\tan \left(\frac{\pi\epsilon}{2} \right) \right) + C_j$$

with some non-zero constants (A_j, B_j, C_j) depending on the remainder.

- The constants $(C_j)_{1 \leq j \leq r}$ and C_0 are the same and may correspond to the normalization issue of the partition function.
- The constants $(A_j, B_j, C_j)_{1 \leq j \leq r}$ and C_0 may be rational numbers.

The case $m = 0$ exhibits an additional symmetry since the number of eigenvalues is precisely a multiple of the number of intervals so that they can spread evenly in all intervals. In particular when evaluating derivatives of the free energies at $\epsilon^* = \frac{1}{2r+1} \mathbf{1}$ cancellations are more likely to happen, thus explaining why the coefficient in front of $\ln \left(\tan \left(\frac{\pi\epsilon}{2} \right) \right)$ vanishes only in this case. However, proving the previous conjecture remains an open problem because computing the spectral curve with arbitrary filling fractions (that we need explicitly in order to compute derivatives of the free energies at ϵ^*) is known to be a hard problem. Nevertheless, we can illustrate our conjecture with numeric simulations:

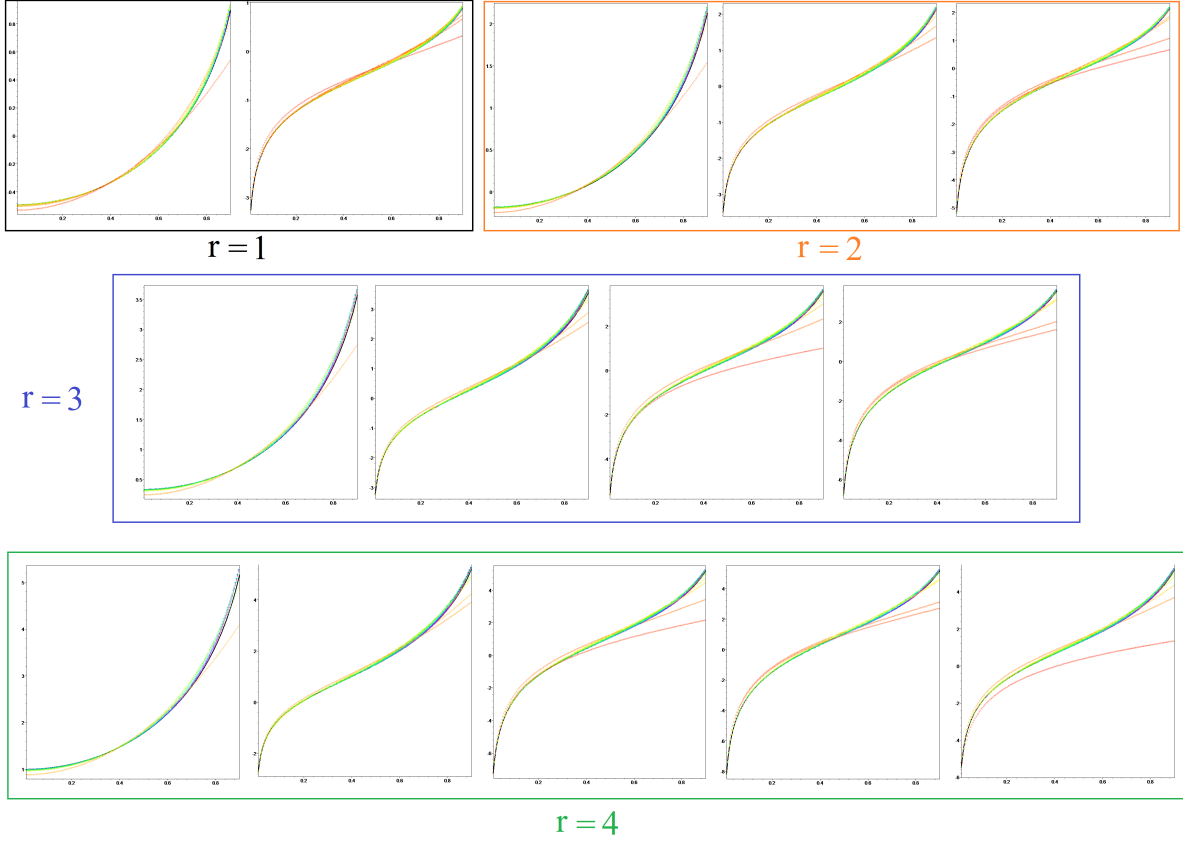


Fig. 6: Representation of $\ln(Z_n(\mathcal{I}_r)) - \frac{1}{2r+1} \ln\left(\sin\left(\frac{\pi\epsilon}{2}\right)\right) + \frac{2r+1}{4} \ln n$ for values of n ranging from 2 to 70 and classified by the remainder m (from left to right: $m = 0$, $m \in \{j, 2r - 1 - j\}$, with j from 1 to r) of the Euclidean division of n by $2r + 1$. The black curves correspond to the best numerical matches of an affine combination of $\ln\left(\cos\left(\frac{\pi\epsilon}{2}\right)\right)$ and $\ln\left(\tan\left(\frac{\pi\epsilon}{2}\right)\right)$ as explained more specifically below.

More precisely, the black curves correspond to the best matches with curves of the form:

$$f(\epsilon) = \alpha \ln\left(\cos\left(\frac{\pi\epsilon}{2}\right)\right) + \beta \ln\left(\tan\left(\frac{\pi\epsilon}{2}\right)\right) + \gamma \quad (3.31)$$

with rational coefficients (α, β, γ) of the form $\frac{i}{256}$ with $i \in \mathbb{Z}$. Note that we chose to express the coefficients as rational numbers with a specific denominator, but we have no evidence that the coefficients are indeed rational or that the denominator is a power of 2. However it seems that this particular choice provides very accurate results and since the topological recursion usually provides rational numbers, it seems a quite legitimate proposal. Numerically, we obtain the best matches for the values:

- Case $r = 1$:

$$\begin{aligned} m = 0 & : (\alpha, \beta, \gamma) = \left(-\frac{3}{4}, 0, -\frac{63}{128}\right) \\ m \in \{1, 2\} & : (\alpha, \beta, \gamma) = \left(-\frac{11}{128}, \frac{85}{128}, -\frac{63}{128}\right) \end{aligned}$$

- Case $r = 2$:

$$m = 0 : (\alpha, \beta, \gamma) = \left(-\frac{5}{4}, 0, -\frac{3}{16}\right)$$

$$\begin{aligned}
m \in \{1, 4\} & : (\alpha, \beta, \gamma) = \left(-\frac{15}{32}, \frac{203}{256}, -\frac{3}{16} \right) \\
m \in \{2, 3\} & : (\alpha, \beta, \gamma) = \left(-\frac{1}{16}, \frac{6}{5}, -\frac{3}{16} \right)
\end{aligned}$$

• Case $r = 3$:

$$\begin{aligned}
m = 0 & : (\alpha, \beta, \gamma) = \left(-\frac{7}{4}, 0, \frac{85}{256} \right) \\
m \in \{1, 6\} & : (\alpha, \beta, \gamma) = \left(-\frac{115}{128}, \frac{109}{128}, \frac{85}{256} \right) \\
m \in \{2, 5\} & : (\alpha, \beta, \gamma) = \left(-\frac{43}{128}, \frac{183}{128}, \frac{85}{256} \right) \\
m \in \{3, 4\} & : (\alpha, \beta, \gamma) = \left(-\frac{7}{128}, \frac{219}{128}, \frac{85}{256} \right)
\end{aligned}$$

• Case $r = 4$:

$$\begin{aligned}
m = 0 & : (\alpha, \beta, \gamma) = \left(-\frac{9}{4}, 0, \frac{255}{256} \right) \\
m \in \{1, 8\} & : (\alpha, \beta, \gamma) = \left(-\frac{355}{256}, \frac{227}{256}, \frac{255}{256} \right) \\
m \in \{2, 7\} & : (\alpha, \beta, \gamma) = \left(-\frac{184}{256}, \frac{398}{256}, \frac{255}{256} \right) \\
m \in \{3, 6\} & : (\alpha, \beta, \gamma) = \left(-\frac{72}{256}, \frac{510}{256}, \frac{255}{256} \right) \\
m \in \{4, 5\} & : (\alpha, \beta, \gamma) = \left(-\frac{17}{256}, \frac{565}{256}, \frac{255}{256} \right)
\end{aligned}$$

Remark 3.2 *The normalization issue in the case of several intervals remains an open problem. Indeed, contrary to the one-cut case, the limiting case $\epsilon \rightarrow 0$ cannot be connected to a Selberg integral. Indeed, if we perform the change of variables:*

$$\mathbf{t} = \tan \left(\frac{\pi r}{2r+1} + \frac{\pi \epsilon}{2(2r+1)} \right) \tilde{\mathbf{t}}$$

then the integral (3.2) does not provide a known integral on $[-1, 1]$ in the limit $\epsilon \rightarrow 0$. In fact, even finding the order γ_n for which $\frac{Z_n(\mathcal{I}_r)}{\epsilon^{\gamma_n}}$ has a non trivial limit is not obvious. Looking at the Toeplitz determinant we conjecture that for all $n \geq 1$:

$$Z_n(\mathcal{I}_r) \underset{\epsilon \rightarrow 0}{=} O(\epsilon^{\gamma_n(r)}) \text{ with } \gamma_n(r) = n - (2r+1) \left\lfloor \frac{n-1}{2r+1} \right\rfloor^2 + (2n-2r-1) \left\lfloor \frac{n-1}{2r+1} \right\rfloor \quad (3.32)$$

In particular, the case $r = 0$ (i.e. the one-cut case) recovers $Z_n(\mathcal{I}_r) \underset{\epsilon \rightarrow 0}{=} O(\epsilon^{n^2})$ as proved in (A.10). However for $r > 0$, the order is lowered to $\epsilon^{\frac{n^2}{2r+1}}$ and tampered by the remainder m of n divided by $2r+1$. From the real or complex integral reformulations, such a complicated behavior at $\epsilon \rightarrow 0$ does not seem obvious to predict and the question of finding the limit seems even more difficult (even the simple fact to know if the limit can be written as a multi-cut real integral with Vandermonde interactions remains open).

3.4 Even number of intervals

The method developed in the last section can be adapted in the case of an even number of intervals. However we need to be careful since in order to apply $\theta \mapsto \tan \frac{\theta}{2}$ we need to avoid the angles $\theta = \pm\pi$. Therefore we use the invariance $\boldsymbol{\theta} \rightarrow \boldsymbol{\theta} + \text{Cste } \mathbf{1}$ of the integral (3.33) to shift the intervals so that they do not contain $\pm\pi$. We define for $s \geq 1$:

$$\begin{aligned}\alpha_k^{(s)} &= \frac{\pi(k - \frac{1}{2})}{s} - \frac{\pi\epsilon}{2s}, \beta_k^{(s)} = \frac{\pi(k - \frac{1}{2})}{s} + \frac{\pi\epsilon}{2s}, \gamma_k^{(s)} = \frac{\pi(k - \frac{1}{2})}{s}, \quad \forall -(s-1) \leq k \leq s \\ \mathcal{I}_s &= \bigcup_{k=-(s-1)}^s [\alpha_k^{(s)}, \beta_k^{(s)}] \\ \mathcal{J}_s &= \bigcup_{k=-(s-1)}^s \left[\tan\left(\frac{\alpha_k^{(s)}}{2}\right), \tan\left(\frac{\beta_k^{(s)}}{2}\right) \right] \\ \mathcal{T}_s &= \{e^{it}, t \in \mathcal{I}_s\}\end{aligned}$$

and the integral:

$$Z_n(\mathcal{I}_s) = \frac{1}{(2\pi)^n n!} \int_{(\mathcal{I}_s)^n} d\theta_1 \dots d\theta_n \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2 \quad (3.33)$$

which is also equal to $\det T_n^{(s)}$ with $(T_n^{(s)})_{i,j} = t_{i-j}$ the $n \times n$ Toeplitz matrix given by:

$$\begin{aligned}t_0 &= \frac{|\mathcal{I}_s|}{2\pi} = \epsilon \\ t_k &= \epsilon \sin_c\left(\frac{k\pi\epsilon}{2s}\right) \delta_{k \equiv 0 [2s]} \text{ for } k \neq 0\end{aligned}$$

Note again that the Toeplitz matrix is mostly empty since only bands with indexes multiple of $2s$ are non-zero. Moreover it is also equal to an Hermitian integral:

$$Z_n(\mathcal{I}_s) = \frac{2^{n^2}}{(2\pi)^n n!} \int_{(\mathcal{J}_s)^n} dt_1 \dots dt_n \Delta(t_1, \dots, t_n)^2 e^{-n \sum_{k=1}^n \ln(1+t_k^2)} \quad (3.34)$$

or a complex integral:

$$Z_n(\mathcal{I}_s) = (-1)^{\frac{n(n+1)}{2}} i^n \int_{(\mathcal{T}_s)^n} du_1 \dots du_n \Delta(u_1, \dots, u_n)^2 e^{-n \sum_{k=1}^n \ln u_k} \quad (3.35)$$

The situation can be illustrated as follow:

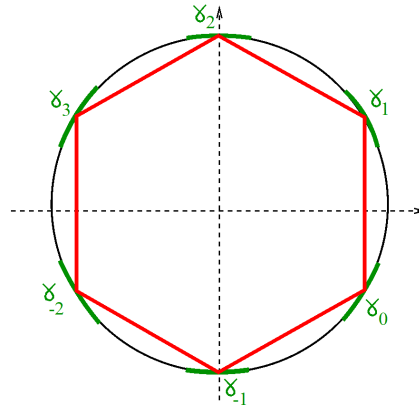


Fig. 7: Illustration (in green) of the set $\mathcal{T}_{s=3}$ for $\epsilon = \frac{1}{5}$.

The method developed in the case of an odd number of intervals can be easily adapted to this case. In particular we can prove the following:

Theorem 3.2 (Results for an even number of symmetric arc-intervals) *In the case of an even number of symmetric arc-intervals (3.33) we have:*

1. *The spectral curve attached to integral (3.33) is given by:*

$$y^2(x) = \frac{\prod_{k=1}^{s-1} \cos^2 \left(\frac{\pi(k-\frac{1}{2})}{2s} \right) x^2 \prod_{k=1}^{s-1} (x^2 - \tan^2 \left(\frac{\pi k}{2s} \right))^2}{\prod_{k=-(s-1)}^s \cos^2 \left(\frac{\pi(k-\frac{1}{2})}{2s} - \frac{\pi\epsilon}{4s} \right) (1+x^2)^2 \prod_{k=-(s-1)}^s \left(x^2 - \tan^2 \left(\frac{\pi(k-\frac{1}{2})}{2s} + \frac{\pi\epsilon}{4s} \right) \right)}$$

2. *The corresponding limiting eigenvalues distribution is given by:*

$$d\mu_\infty(x) = \frac{\prod_{k=1}^{s-1} \cos \left(\frac{\pi(k-\frac{1}{2})}{2s} \right)}{\prod_{k=-(s-1)}^s \cos \left(\frac{\pi(k-\frac{1}{2})}{2s} - \frac{\pi\epsilon}{4s} \right) \pi(1+x^2) \prod_{k=-(s-1)}^s \sqrt{\left| x^2 - \tan^2 \left(\frac{\pi(k-\frac{1}{2})}{2s} + \frac{\pi\epsilon}{4s} \right) \right|}} |x| \prod_{k=1}^{s-1} |x^2 - \tan^2 \left(\frac{\pi k}{2s} \right)| \mathbb{1}_{\mathcal{I}_s}(x) dx$$

3. *The filling fractions are the same in all the $2s$ intervals:*

$$\forall - (s-1) \leq k \leq s : \epsilon_{k+s} \stackrel{\text{def}}{=} \int_{\tan \left(\frac{\pi(k-\frac{1}{2})}{2s} - \frac{\pi\epsilon}{4s} \right)}^{\tan \left(\frac{\pi(k-\frac{1}{2})}{2s} + \frac{\pi\epsilon}{4s} \right)} d\mu_\infty(x) = \frac{1}{2s}$$

4. *The function $\ln Z_n(\mathcal{I}_s)$ admits a large n expansion given by theorem 3.1 with partial reconstruction by the topological recursion given by proposition 3.2. In particular we have for $s \geq 1$:*

$$\ln(Z_n(\mathcal{I}_s)) = \frac{1}{2s} \ln \left(\sin \left(\frac{\pi\epsilon}{2} \right) \right) - \frac{2s}{4} \ln n + O(1)$$

The $O(1)$ order depends on the remainder m of n modulo $2s$. Numerically, its dependence in ϵ follows the conjecture 3.1 (with $2r+1$ replaced by $2s$ and the special case still given by $m=0$).

5. *In the limit $\epsilon \rightarrow 0$, we conjecture that:*

$$Z_n(\mathcal{I}_s) \underset{\epsilon \rightarrow 0}{=} O(\epsilon^{\gamma_n(s)}) \text{ with } \gamma_n(s) = n - 2s \left\lfloor \frac{n-1}{2s} \right\rfloor^2 + 2(n-s) \left\lfloor \frac{n-1}{2s} \right\rfloor$$

4 Number of intervals scaling with n

The correspondence with Toeplitz determinants allows the study of integrals (1.1) in the peculiar situation where the symbol is a indicator function of a infinite number of intervals in the limit $n \rightarrow \infty$. Indeed, in the previous sections, we studied the large n limit in the case of a fixed number of intervals, but nothing prevents us from scaling the number of intervals with n in the definition. In general, studying such situations is complicated but it turns out that the correspondence with Toeplitz determinants provides, in specific cases, some explicit results. Let us define:

Definition 4.1 Let $\epsilon > 0$ be a given real number and let $g : \mathbb{N} \rightarrow \mathbb{N}$ be non-decreasing function with $\lim_{n \rightarrow \infty} g(n) = +\infty$. We define the following integral:

$$Z_n(g) = \frac{1}{(2\pi)^n n!} \int_{(\mathcal{I}_n(g))^n} d\theta_1 \dots d\theta_n \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^2 \quad (4.1)$$

where $\mathcal{I}_n(g)$ is a union of $N = g(n)$ intervals in $[-\pi, \pi]$ given by:

$$\begin{aligned} \mathcal{I}_n(g) &= \bigcup_{k=-r}^r \left[\frac{2\pi k}{2r+1} - \frac{\pi\epsilon}{2r+1}, \frac{2\pi k}{2r+1} + \frac{\pi\epsilon}{2r+1} \right] \text{ if } N = 2r+1 \text{ is odd} \\ \mathcal{I}_n(g) &= \bigcup_{k=-(s-1)}^s \left[\frac{\pi(k - \frac{1}{2})}{s} - \frac{\pi\epsilon}{2s}, \frac{\pi(k - \frac{1}{2})}{s} + \frac{\pi\epsilon}{2s} \right] \text{ if } N = 2s \text{ is even} \end{aligned} \quad (4.2)$$

In other words, we have $N = g(n)$ disjoint intervals of size $\frac{2\pi\epsilon}{N}$ distributed uniformly on the unit circle around each of the N^{th} root of unity (with a conventional shift by $-\frac{\pi\epsilon}{N}$ in the even case that does not change the value of integral):

$$Z_n(g) = (-1)^{\frac{n(n+1)}{2}} i^n \int_{(\mathcal{T}_n(g))^n} du_1 \dots du_n \Delta(u_1, \dots, u_n)^2 e^{-n \sum_{k=1}^n \ln u_k}$$

with $\mathcal{T}_n(g) = \{e^{it}, t \in \mathcal{I}_n(g)\}$

Note that the notion of spectral curve and thus the topological recursion does not seem to apply directly to this setting. Indeed, as presented in appendix B, the limiting eigenvalues distribution attached to a spectral curve may only contain a finite number of intervals (one more than the genus of the Riemann surface defining the spectral curve). On the contrary, in our situation, the number of intervals increases with n and tends to infinity in the limit $n \rightarrow \infty$. Eventually integral (4.1) may also be written in terms of a real integral:

$$Z_n(g) = \frac{2^{n^2}}{(2\pi)^n n!} \int_{(\mathcal{J}_n(g))^n} dt_1 \dots dt_n \Delta(t_1, \dots, t_n)^2 e^{-n \sum_{k=1}^n \ln(1+t_k^2)} \quad (4.3)$$

with

$$\begin{aligned} \mathcal{J}_n(g) &= \bigcup_{k=-r}^r \left[\tan\left(\frac{\pi k}{2r+1} - \frac{\pi\epsilon}{2(2r+1)}\right), \tan\left(\frac{\pi k}{2r+1} + \frac{\pi\epsilon}{2(2r+1)}\right) \right] \\ &\quad \text{if } N = g(n) = 2r+1 \text{ is odd} \\ \mathcal{J}_n(g) &= \bigcup_{k=-(s-1)}^s \left[\tan\left(\frac{\pi(k - \frac{1}{2})}{2s} - \frac{\pi\epsilon}{4s}\right), \tan\left(\frac{\pi(k - \frac{1}{2})}{2s} + \frac{\pi\epsilon}{4s}\right) \right] \\ &\quad \text{if } N = g(n) = 2s \text{ is even} \end{aligned}$$

Heuristically, the real integral reformulation seems to indicate that three different situations may arise:

1. The number of intervals grows similarly or faster than the number of eigenvalues (for example $g(n) = \lfloor n^\alpha \rfloor$ with $\alpha \geq 1$). In that case, for a fixed n , since the potential is not strong enough to confine the eigenvalues compared to the Vandermonde repulsion, it seems reasonable that each eigenvalue tends to occupy an empty interval. If too many intervals are available (for example $g(n) = \lfloor n^\alpha \rfloor$ with $\alpha > 1$) then by symmetry the configurations with the lowest energy are those where the eigenvalues only occupy n intervals corresponding to the n^{th} (and not N^{th}) roots of unity.
2. The number of intervals grows slower than the number of eigenvalues (for example $g(n) = \lfloor n^\alpha \rfloor$ with $\alpha < 1$). In that case, an infinite number of eigenvalues share each interval. By symmetry, they occupy each interval in the same way but it is not clear what the configuration with the lowest energy might be or if a limiting density of eigenvalues exist on each interval.
3. The number of intervals remains strictly lower than the number of eigenvalues but grows proportionally to n . This situation happens when $g(n) = n - m$ for a given integer $m \geq 1$ or in the case of $g(n) = \left\lfloor \frac{p}{q}n \right\rfloor$ with $\frac{p}{q}$ a given rational number with $0 < \frac{p}{q} < 1$. In that case only a finite number of eigenvalues share each interval but it is not obvious to determine if a limiting eigenvalues density exists.

Consequently, determining the limiting eigenvalues distribution (if it exists) seems a rather difficult problem. On the other hand, if one is only interested in the exact value of the partition function, it is more efficient to use the correspondence with Toeplitz determinants given by:

$$\begin{aligned}
Z_n(g) &= \det T_n^{(g)} \text{ with } (T_n^{(g)})_{i,j} = t_{i-j} \text{ the } n \times n \text{ Toeplitz matrix given by:} \\
t_0 &= \epsilon \\
t_k &= \epsilon \sin_c \left(\frac{k\pi\epsilon}{N} \right) \delta_{k \equiv 0 [N]} \text{ for } k \neq 0 \text{ where } N = g(n)
\end{aligned} \tag{4.4}$$

In particular, we obtain:

- For values of n satisfying $g(n) \geq n$ then we have:

$$Z_n(g) = \epsilon^n = \left(\frac{|\mathcal{I}_n(g)|}{2\pi} \right)^n \tag{4.5}$$

The result does not depend on the precise expression of the function g (for example it is independent of $\alpha \geq 1$ if we take $g(n) = \lfloor n^\alpha \rfloor$). In particular, $\ln Z_n(g) = n \ln \epsilon$ and if there exists $n_0 > 0$ such that $n \geq n_0 \Rightarrow g(n) \geq n$ then $\ln Z_n(g)$ admit a large n expansion (given by $n \ln \epsilon$) which is not of the form proposed in theorem 3.1. We also note that the value of $Z_n(g)$ is exactly the same as:

$$Z_n(g) = \frac{1}{(2\pi)^n n!} \int_{(\mathcal{I}_n(g))^n} (n!) d\theta_1 \dots d\theta_n$$

In other words, it seems that the interaction $|\Delta(e^{i\theta_1}, \dots, e^{i\theta_n})|^2$ is useless and only provides a factor $n!$.

- If $g(n) = n - 1$, then a direct computation of the Toeplitz determinant gives $Z_n(g) = \epsilon^n (1 - \sin_c^2(\pi\epsilon))$. More generally, if $g(n) = n - s$ with $1 \leq s \leq n - 1$ a given integer (independent of n), then

$$Z_n(g) = \epsilon^n (1 - \sin_c^2(\pi\epsilon))^s \tag{4.6}$$

- In the case $g = \lfloor \frac{n}{2} \rfloor$, straightforward determinant computations show that:

$$Z_n(g) = \epsilon^n (1 - \sin_c^2(\pi\epsilon))^{\lfloor \frac{n}{2} \rfloor} \begin{cases} 1 & \text{if } n \equiv 0 \pmod{2} \\ (\sin_c(2\pi\epsilon) - 1)(2\sin_c^2(\pi\epsilon) - 1 - \sin_c(2\pi\epsilon)) & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

More generally, if $g = \lfloor \frac{n}{m} \rfloor$ for a given integer m (independent of n), then $Z_n(g)$ should exhibit similar expressions with a factor ϵ^n , a term to the power $\lfloor \frac{n}{m} \rfloor$ and a term depending on the value of n modulo m . Again such cases do not have a large n expansion given by theorem 3.1.

- When $g(n) \underset{n \rightarrow \infty}{=} o(n)$, the situation is much more complicated. Indeed, in that case all values of $\sin_c(k\pi\epsilon)$ with $k \geq 1$ appear in the computation of the determinant. To our knowledge, the value of the partition function is not known nor the limiting eigenvalues density if it exists.

5 Conclusion and outlooks

In this article, we proposed a rigorous mathematical derivation of the asymptotic expansions of Toeplitz determinants with symbols given by $f = \mathbb{1}_{\mathcal{T}(\alpha, \beta)}$ where $\mathcal{T}(\alpha, \beta) = \{e^{it}, t \in [\alpha, \beta]\}$. This generalizes Widom's result and completes the approach developed in [4]. We also provided numerical simulations up to $o(\frac{1}{n^4})$ to illustrate these results. For symbols $f = \mathbb{1}_{\mathcal{T}_d}$ with $\mathcal{T}_d = \bigcup_{k=1}^d [\alpha_k, \beta_k]$ and $d \geq 2$, the situation is more complex, but we were able to provide a rigorous derivation of the large n asymptotic of the corresponding Toeplitz determinants when the arc-intervals exhibit a discrete rotational symmetry on the unit circle. We also provided the first terms of the large n expansion up to $O(1)$ and we proposed a conjecture for the $O(1)$ term supported by numeric simulations. Moreover, the results presented in this article raise the following challenges:

- Prove conjecture 3.1 regarding the $O(1)$ term in the symmetric multi-cut case. In particular, it would be interesting to find a way to obtain the normalization constants. This requires to connect the Toeplitz integral to a known integral for at least one value of the parameter ϵ (very likely $\epsilon \rightarrow 0$).
- Compute explicitly the free energies $(F^{(g)})_{g \geq 0}$ associated to the spectral curve $y = \frac{1}{\sqrt{x^2 - 1}}$ to get the normalizing constants $(F^{(g)}(a = 0))_{g \geq 0}$ in the one-cut case (2.3). To our knowledge, such results are available for the normalized spectral curve with two soft edges: $y = 2\sqrt{x^2 - 1}$ and for the normalized spectral curve with one soft edge and one hard edge: $y = 2\sqrt{\frac{x-1}{x}}$, but are missing for the normalized two hard edges case.
- In the multi-cut cases without discrete symmetry, it would be interesting to prove that, for generic choices of the edges, the corresponding spectral curves are regular and that the hypothesis required to prove the large n asymptotic of theorem (3.1) are verified. Contrary to the case with a discrete rotational symmetry, it seems unlikely that we obtain an explicit formula for the spectral curves. However it may be possible to obtain sufficient information on the location of the zeros of $y(x)$ in order to prove that the spectral curves are regular.

- The method developed in this article could also be tried for more general Toeplitz determinants. In particular, the transition from a symbol supported on a compact set of $\{e^{it}, t \in (-\pi, \pi)\}$ to a strictly positive symbol on the unit circle deserves some analysis. Indeed, in the case of a strictly positive symbol on the unit circle, Szegő's theorem implies that $\frac{1}{n} \ln \det T_n$ admits a non-trivial limit (given by $\frac{1}{2\pi} \int_0^{2\pi} \ln(f(e^{i\theta})) d\theta$) so that the large n expansion proposed in theorem 3.1 requires some adaptations. However, it may happen that the sub-leading corrections may still be given by theorem 3.1 and in particular that sub-leading corrections may be reconstructed from the topological recursion. This conjecture is supported by the fact that Toeplitz determinants can also be reformulated as Fredholm determinants (See [30]) that are known to be deeply related with the topological recursion.

Acknowledgments

I would like to thank A. Guionnet that indirectly gave me the idea to review in a rigorous way the tools and theorems used in this article. I also would like G. Borot for very fruitful discussions about the normalizing constants. Eventually, I would like to thank Université de Lyon, Université Jean Monnet and Institut Camille Jordan for financial and material support.

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A Computation of $F_{\text{Top.Rec}}^{(0)}$, $F_{\text{Top.Rec}}^{(1)}$ and of the normalizing constants in the one interval case

A.1 Computation of $F_{\text{Top.Rec}}^{(0)}$

We want to compute the first free energy of the curve:

$$\begin{aligned} x(z) &= \frac{1}{2} \tan\left(\frac{\pi\epsilon}{2}\right) \left(z + \frac{1}{z}\right) \\ y(z) &= \frac{1}{\sin(\frac{\pi\epsilon}{2}) \left(1 + \frac{1}{4} \tan^2(\frac{\pi\epsilon}{2}) \left(z + \frac{1}{z}\right)\right) \left(z - \frac{1}{z}\right)} \end{aligned} \quad (\text{A.1})$$

We first observe that the one-form ydx is given by:

$$ydx(z) = \left(\frac{t_1}{z - Z_1} + \frac{t_2}{z - Z_2} + \frac{t_3}{z - Z_3} + \frac{t_4}{z - Z_4} \right) dz \quad (\text{A.2})$$

with:

$$\begin{aligned} Z_1 &= i \frac{1 - \cos(\frac{\pi\epsilon}{2})}{\sin(\frac{\pi\epsilon}{2})} \text{ and } t_1 = \frac{1}{2} \\ Z_2 &= i \frac{1 + \cos(\frac{\pi\epsilon}{2})}{\sin(\frac{\pi\epsilon}{2})} \text{ and } t_2 = -\frac{1}{2} \\ Z_3 &= -i \frac{1 + \cos(\frac{\pi\epsilon}{2})}{\sin(\frac{\pi\epsilon}{2})} \text{ and } t_3 = -\frac{1}{2} \\ Z_4 &= -i \frac{1 - \cos(\frac{\pi\epsilon}{2})}{\sin(\frac{\pi\epsilon}{2})} \text{ and } t_4 = \frac{1}{2} \end{aligned} \quad (\text{A.3})$$

Thus the one form ydx has 4 simple poles that are not branchpoints but only poles of $y(z)$. The local coordinate around each point is given by:

$$x_k(z) = \frac{1}{x(z) - x(Z_k)} = \frac{2z}{\tan(\frac{\pi\epsilon}{2})(z - Z_k)(z - \frac{1}{Z_k})}$$

Hence the local potential:

$$V_k(p) = \operatorname{Res}_{q \rightarrow Z_k} y dx(q) \ln \left(1 - \frac{x(q) - x(Z_1)}{x(p) - x(Z_1)} \right)$$

is trivially vanishing in all four cases since the poles are simple. Eventually we end up with the computation of:

$$\mu_k = \left(\int_{Z_k}^o y dx(z) - t_k \frac{dx(z)}{x(z) - x(Z_k)} \right) + t_k \ln(x(r) - x(Z_k)) \quad (\text{A.4})$$

Observing that $\frac{x'(z)}{x(z) - x(Z_k)} = -\frac{1}{z} + \frac{1}{z - Z_k} + \frac{1}{z - \frac{1}{Z_k}}$ we have:

$$\begin{aligned} \mu_k &= \sum_{j \neq k} t_j \ln(o - Z_j) - \sum_{j \neq k} t_j \ln(Z_k - Z_j) + t_k \ln \left(1 - \frac{1}{Z_k^2} \right) + \frac{t_k}{2} \tan \left(\frac{\pi \epsilon}{2} \right) \\ &= \sum_{j \neq k} t_j \ln(o - Z_j) - \sum_{j \neq k} t_j \ln(Z_k - Z_j) + t_k \ln \left(1 - \frac{1}{Z_k^2} \right) + \frac{t_k}{2} \tan \left(\frac{\pi \epsilon}{2} \right) \end{aligned} \quad (\text{A.5})$$

Hence in the end, observing that $\sum_{k=1}^4 t_j = 0$ we get that the dependence in o vanishes (as claimed in [18]) and we find:

$$\begin{aligned} F_{\text{Top.Rec.}}^{(0)} &\stackrel{\text{def}}{=} \frac{1}{2} \sum_{k=1}^4 t_k \mu_k \\ &= -\frac{1}{2} \sum_{k=1}^4 \sum_{j \neq k} t_k t_j \ln(Z_k - Z_j) + \frac{1}{2} \sum_{k=1}^4 t_k \ln \left(1 - \frac{1}{Z_k^2} \right) + \frac{1}{2} \tan \left(\frac{\pi \epsilon}{2} \right) \sum_{k=1}^4 t_k^2 \\ &= -\frac{1}{2} \sum_{j < k=1}^4 t_k t_j \ln(-(Z_k - Z_j)^2) + \frac{1}{2} \sum_{k=1}^4 t_k \ln \left(1 - \frac{1}{Z_k^2} \right) + \frac{1}{2} \tan \left(\frac{\pi \epsilon}{2} \right) \sum_{k=1}^4 t_k^2 \\ &= \ln 2 - \ln \left(\sin \frac{\pi \epsilon}{2} \right) \end{aligned} \quad (\text{A.6})$$

A.2 Computation of $F_{\text{Top.Rec.}}^{(1)}$

The computation of $F_{\text{Top.Rec.}}^{(1)}$ in the case of two hard edges is relatively straightforward. First, we observe that we have:

$$W_1^{(1)}(z) = \frac{1}{2 \cos \left(\frac{\pi \epsilon}{2} \right) (z - 1)^2 (z + 1)^2} \quad (\text{A.7})$$

Then the computation of $F^{(1)}$ can be performed using the formalism of [18] (in which case the Bergmann tau-function is $\ln \tau_B = \frac{3}{8} \ln \left(\cos \left(\frac{\pi \epsilon}{2} \right) \right)$) or with the refined formalism of [24]. In both cases, the computations are straightforward since the spectral curve only has two branchpoints at $z = \pm 1$. Eventually we find:

$$F_{\text{Top.Rec.}}^{(1)} = \frac{1}{4} \ln \left(\cos \left(\frac{\pi \epsilon}{2} \right) \right) = -\frac{1}{8} \ln \left(1 + \tan^2 \left(\frac{\pi \epsilon}{2} \right) \right) \quad (\text{A.8})$$

We note that our result differs from the one presented in [4] in which the Bergmann tau-function is incorrect.

A.3 Normalization with a Selberg integral

We have the following Selberg integral:

$$\begin{aligned}
S_n(1, 1, 1) &= \int_{[-1,1]^n} \prod_{1 \leq i < j \leq n} (u_i - u_j)^2 du_1 \dots du_n = \frac{2^{n^2}}{n!} \prod_{j=1}^{n-1} \frac{\Gamma^2(j+1)\Gamma(j+2)}{\Gamma(n+j+1)\Gamma(2)} \\
&= \frac{2^{n^2} (n!) \left(\prod_{j=1}^{n-1} j! \right)^4}{\prod_{j=1}^{2n-1} j!}
\end{aligned} \tag{A.9}$$

We get:

$$\begin{aligned}
Z_n(a) &= \frac{2^{n^2}}{(2\pi)^n n!} \int_{[-a,a]^n} \Delta(t_1, \dots, t_n)^2 e^{-n \sum_{i=1}^n \ln(1+t_i^2)} dt_1 \dots dt_n \\
&= \frac{a^{n^2} 2^{n^2}}{(2\pi)^n n!} \int_{[-1,1]^n} \Delta(u_1, \dots, u_n)^2 e^{-n \sum_{i=1}^n \ln(1+a^2 u_i^2)} du_1 \dots du_n \\
&\stackrel{\text{def}}{=} \frac{a^{n^2} 2^{n^2}}{(2\pi)^n n!} S_a
\end{aligned} \tag{A.10}$$

with:

$$S_a = \int_{[-1,1]^n} \Delta(u_1, \dots, u_n)^2 e^{-n \sum_{i=1}^n \ln(1+a^2 u_i^2)} du_1 \dots du_n \tag{A.11}$$

S_a is an Hermitian matrix integral on $I = [-1, 1]$ with potential $V_a(x) = \ln(1 + a^2 x^2)$. It is continuous in a and in particular for $a = 0$ we find:

$$S_0 = S_n(1, 1, 1) = \frac{2^{n^2} (n!) \left(\prod_{j=1}^{n-1} j! \right)^4}{\prod_{j=1}^{2n-1} j!} \tag{A.12}$$

Therefore we should have:

$$\begin{aligned}
\ln Z_n(a) - n^2 \ln(a) &\xrightarrow{a \rightarrow 0} \ln(S_n(1, 1, 1)) + 2n^2 \ln 2 - n \ln(2\pi) - \ln(n!) \\
&= 4 \ln(G(n+1)) - \ln(G(2n+1)) + 2n^2 \ln 2 - n \ln(2\pi)
\end{aligned} \tag{A.13}$$

where the function $G(z)$ is the G -Barnes function whose asymptotic expansion is for $N \in \mathbb{N}^*$:

$$\begin{aligned}
\ln(G(N+1)) &= \ln \left(\prod_{i=1}^{N-1} i! \right) \\
&= \frac{N^2}{2} \ln N + \frac{N}{2} \ln(2\pi) - \frac{1}{12} \ln N + \xi'(-1) - \frac{3}{4} N^2 + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g}
\end{aligned} \tag{A.14}$$

Using this asymptotic for $N = n$ and $N = 2n$ as well as Stirling's formula:

$$\ln(n!) = \frac{1}{2} \ln(2\pi) + n \ln n + \frac{1}{2} \ln n - n + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)n^{2k-1}} \tag{A.15}$$

we can compute the asymptotic expansion of the r.h.s. of (A.13). We obtain:

$$\begin{aligned} & \ln(S_n(1, 1, 1)) + 2n^2 \ln 2 - n \ln(2\pi) - \ln(n!) \\ &= -\frac{1}{4} \ln n + 3\xi'(-1) + \frac{1}{12} \ln 2 + \sum_{g=1}^{\infty} \frac{4(1 - 2^{-2g-2})B_{2g+2}}{2g(2g+2)n^{2g}} \end{aligned} \quad (\text{A.16})$$

On the other side of (A.13) we have:

$$\begin{aligned} \ln Z_n(a) - n^2 \ln \left(\tan \frac{\pi\epsilon}{2} \right) &\xrightarrow{\epsilon \rightarrow 0} (f^{\{-2\}} - \ln 2)n^2 - \frac{1}{4} \ln n + \sum_{k=-1}^{\infty} f^{\{2k+1\}} n^{-2k-1} \\ &+ \sum_{g=0}^{\infty} (-F^{(g+1)}(\epsilon=0) + f^{\{2g\}}) n^{-2g} \end{aligned} \quad (\text{A.17})$$

We observe that the series expansions (A.16) and (A.17) are compatible and we get:

$$\begin{aligned} f^{\{-2\}} &= \ln 2 \\ f^{\{-1\}} &= 0 \\ f^{\{0\}} &= F^{(1)}(a=0) + 3\xi'(-1) + \frac{1}{12} \ln 2 = 3\xi'(-1) + \frac{1}{12} \ln 2 \\ f^{\{2g\}} &= F^{(g+1)}(a=0) + \frac{4(1 - 2^{-2g-2})B_{2g+2}}{2g(2g+2)} \text{ for } g \geq 1 \\ f^{\{2k+1\}} &= 0 \text{ for } k \geq 0 \end{aligned} \quad (\text{A.18})$$

In other words, we finally obtain with $a = \tan \frac{\pi\epsilon}{2}$:

$$\begin{aligned} \ln Z_n(a) &= n^2 \ln \left(\sin \left(\frac{\pi\epsilon}{2} \right) \right) - \frac{1}{4} \ln n - \frac{1}{4} \ln \left(\cos \left(\frac{\pi\epsilon}{2} \right) \right) + 3\xi'(-1) + \frac{1}{12} \ln 2 \\ &+ \sum_{g=1}^{\infty} \left(F^{(g+1)}(a=0) - F^{(g+1)}(a) + \frac{4(1 - 2^{-2g-2})B_{2g+2}}{2g(2g+2)} \right) n^{-2g} \end{aligned} \quad (\text{A.19})$$

The first orders are:

$$\begin{aligned} \ln Z_n(a) &= n^2 \ln \left(\sin \left(\frac{\pi\epsilon}{2} \right) \right) - \frac{1}{4} \ln n - \frac{1}{4} \ln \left(\cos \left(\frac{\pi\epsilon}{2} \right) \right) + 3\xi'(-1) + \frac{1}{12} \ln 2 \\ &+ \frac{1}{64n^2} \left(2 \tan^2 \left(\frac{\pi\epsilon}{2} \right) - 1 \right) + \frac{1}{256n^4} \left(1 + 2 \tan^2 \left(\frac{\pi\epsilon}{2} \right) + 10 \tan^4 \left(\frac{\pi\epsilon}{2} \right) \right) \\ &+ O \left(\frac{1}{n^6} \right) \end{aligned} \quad (\text{A.20})$$

For $g \geq 2$, we can compute the $F_{\text{Top.Rec.}}^{(g)}(a=0)$ using the symplectic invariance of the free energies. Indeed, the symplectic change of variables $(\tilde{x}, \tilde{y}) = (\frac{x}{\sin \frac{\pi\epsilon}{2}}, y \sin \frac{\pi\epsilon}{2})$ in (2.28) provides a new spectral curve with the same symplectic invariants for $g \geq 2$. This new spectral curve has a regular limit when $a \rightarrow 0$ and since from [18] we know that the free energies $F^{(g)}$ (with $g \geq 2$) depend continuously on regular variations of the curve, we end up with:

$$\forall g \geq 2 : F_{\text{Top.Rec.}}^{(g)}(a=0) = \tilde{F}^{(g)} : \text{Free energies of the curve } y^2 = \frac{1}{x^2 - 1} \quad (\text{A.21})$$

The last curve is particularly easy to handle since it can be parametrized globally on $\mathbb{C} \cup \{\infty\}$ with:

$$x(z) = \frac{1}{2} \left(z + \frac{1}{z} \right) \quad \text{and} \quad y(z) = \frac{2z}{z^2 - 1} \quad (\text{A.22})$$

The one-form $ydx(z) = \frac{dz}{z}$ is very simple but unfortunately the explicit values of the free energies $\left(\tilde{F}^{(g)}\right)_{g \geq 0}$ are not currently known. However, it is likely that they could be obtained from the asymptotic expansion of the Jacobi polynomials or from some enumerative problems in enumerative geometry (like Hurwitz numbers, Gromov-Witten invariants, etc.).

B The Eynard-Orantin topological recursion

In this section we briefly review the formalism of the topological recursion as presented in [18]. More general versions of the topological recursion can be found in the literature but we restrict ourselves to the original simpler version of [18] that is sufficient for the purposes of this article. Let us start by the definition of a spectral curve:

Definition B.1 (Spectral curve, branchpoints, normalized bi-differential) *A spectral curve is the data of two meromorphic functions $(x(z), y(z))$ on a Riemann surface Σ of genus \mathfrak{g} . This is equivalent to the data of a polynomial P such that $P(x, y) = 0$ and therefore to an algebraic equation between x and y . When the genus \mathfrak{g} of Σ is strictly positive, we complete the data of the spectral curve with the choice of a basis of homology cycles $(\mathcal{A}_i, \mathcal{B}_i)_{1 \leq i \leq \mathfrak{g}}$ such that:*

$$\begin{aligned} \forall i \neq j & : \mathcal{A}_i \cap \mathcal{A}_j = \mathcal{B}_i \cap \mathcal{B}_j = \emptyset \\ \forall 1 \leq i, j \leq \mathfrak{g} & : \mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j} \end{aligned}$$

Then it follows from standard results of algebraic geometry that there exists a unique symmetric bi-differential $B(z_1, z_2)$ (sometimes called “Bergmann kernel”) such that:

- B is holomorphic on $\Sigma \times \Sigma$ except at coinciding points where it behaves like:

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{regular}_{z_1 \rightarrow z_2}$$

- B is normalized on the basis of cycles $(\mathcal{A}_i, \mathcal{B}_i)_{1 \leq i \leq \mathfrak{g}}$ in the following way:

$$\oint_{\mathcal{A}_i} B(z_1, z_2) = 0 \text{ for all } 1 \leq i \leq \mathfrak{g}$$

The branchpoints $(a_i)_{1 \leq i \leq R}$ (with $R \geq 1$) of the spectral curve are the points where dx vanishes. The spectral curve is said “regular” if the branchpoints are simple zeros of dx . When the spectral curve is regular, we can define locally around each branchpoint an involution $z \mapsto \bar{z}$ such that $x(z) = x(\bar{z})$.

In this paper we will only restrict ourselves to the case of regular spectral curves since the situation is much more complicated when the curve is not regular. We remark that when the spectral curve is regular and of genus 0, then there exists a global parametrization $(x(z), y(z))$ with $z \in \mathbb{C} \cup \{\infty\}$ of the spectral curve. Moreover, the involution $z \mapsto \bar{z}$ is defined globally on the spectral curve and the normalized bi-differential B is explicitly given by $B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$. We now have all the ingredients to define the correlators and free energies associated to a spectral curve.

Definition B.2 (Definition 4.2 of [18]) For $g \geq 0$ and $n \geq 1$, the Eynard-Orantin differentials (known also as “correlation functions” or “correlators”) $\omega_n^{(g)}(z_1, \dots, z_n)$ of type (g, n) associated to the spectral curve $(x(z), y(z))$ are defined by the following recursive relations:

$$\omega_1^{(0)}(z_1) = (y(z_1) - y(\bar{z}_1))dx(z_1) \quad (\text{B.1})$$

$$\omega_2^{(0)}(z_1, z_2) = B(z_1, z_2), \quad (\text{B.2})$$

$$\begin{aligned} \omega_{n+1}^{(g)}(z_0, z_1, \dots, z_n) &= \sum_{i=1}^R \text{Res}_{z \rightarrow a_i} K(z_0, z) \left[\omega_{n+1}^{(g-1)}(z, \bar{z}, z_1, \dots, z_n) \right. \\ &\quad \left. + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1, \dots, n\}}} \omega_{1+|I|}^{(g_1)}(z, z_I) \omega_{1+|J|}^{(g_2)}(\bar{z}, z_J) \right]. \end{aligned} \quad (\text{B.3})$$

Here

$$K(z_0, z) = \frac{\int_z^{\bar{z}} \omega_2^{(0)}(\cdot, z_0)}{(y(z) - y(\bar{z}))dx(z)} \quad (\text{B.4})$$

is called the recursion kernel, and the ‘ $'$ in the last line of (B.1) means that the cases $(g_1, I) = (0, \emptyset)$ and $(g_2, J) = (0, \emptyset)$ must be excluded from the sum.

The Eynard-Orantin differentials $\omega_n^{(g)}$ ’s are meromorphic multi-differentials on Σ^n and are known to be holomorphic except at the branchpoints if $(g, n) \neq (0, 1), (0, 2)$. In [18], the authors also introduced free energies (also called “symplectic invariants”) $(F^{(g)})_{g \geq 0}$ defined by:

Definition B.3 (Definition 4.3 of [18]) The g^{th} symplectic invariant $F^{(g)}$ associated to the spectral curve $(x(z), y(z))$ is defined by:

$$F^{(g)} = \frac{1}{2-2g} \sum_{i=1}^R \text{Res}_{z \rightarrow a_i} \Phi(z) \omega_1^{(g)}(z) \quad \text{for } g \geq 2$$

where

$$\Phi(z) = \int_{z_o}^z y(\tilde{z})dx(\tilde{z}) \quad (z_o \text{ is any generic point}).$$

$F^{(0)}$ and $F^{(1)}$ are defined with specific formulas that can be found in [18].

Note that this definition extends to the case $n = 0$ (with the identification $\omega_0^{(g)} = F^{(g)}$) the following property:

$$\omega_n^{(g)}(z_1, \dots, z_n) = \frac{1}{2-2g-n} \sum_{i=1}^R \text{Res}_{z \rightarrow a_i} \Phi(z) \omega_{n+1}^{(g)}(z, z_1, \dots, z_n) \quad \text{for } g \geq 0 \text{ and } n \geq 0$$

Note also that the Eynard-Orantin differentials or the symplectic invariants do not depend on the choice of parametrization $(x(z), y(z))$. Eventually as suggested by their name, the symplectic invariants $(F^{(g)})_{g \geq 0}$ are invariant under transformations of the spectral curve $(x, y) \rightarrow (\tilde{x}, \tilde{y})$ such that $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$, i.e. transformations that preserve the symplectic form $dx \wedge dy$. This property does not hold in general for the Eynard-Orantin differentials $\omega_n^{(g)}$.